Filtration Simplification for Persistent Homology via Edge Contraction

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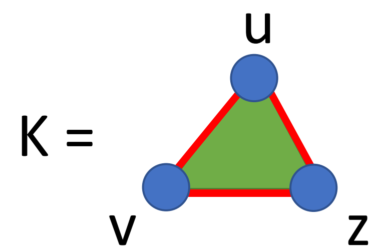
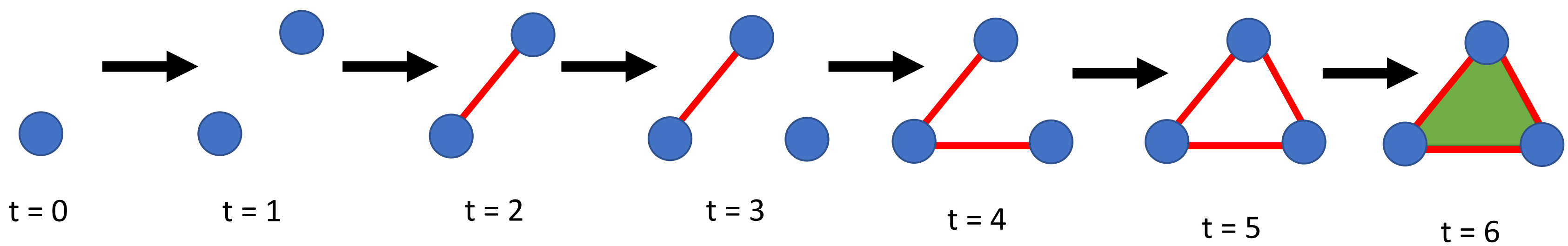
Persistent Homology



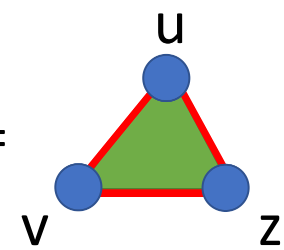


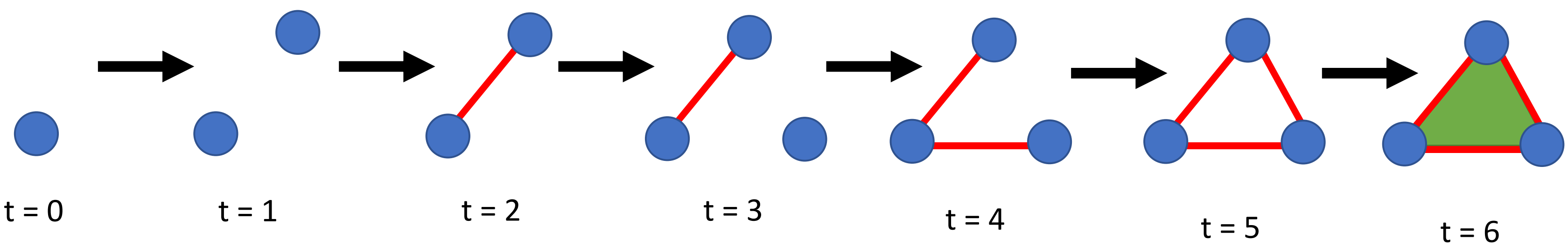
{u}:{u,v}
{z}:{v,z}
{u,z}:{u,v,z}

$$K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6$$

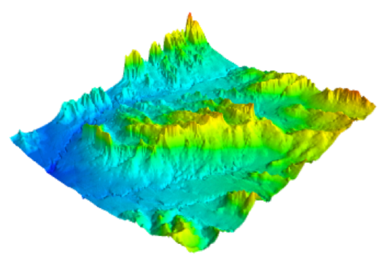
Persistence Diagrams

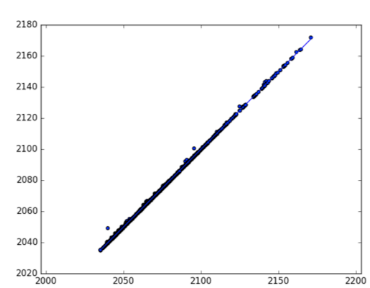
K =

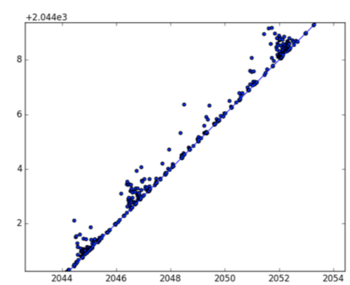




{u}:{u,v} => point at (1,2)
{z}:{v,z} => point at (3,4)
{u,z}:{u,v,z} => point at (5,6)



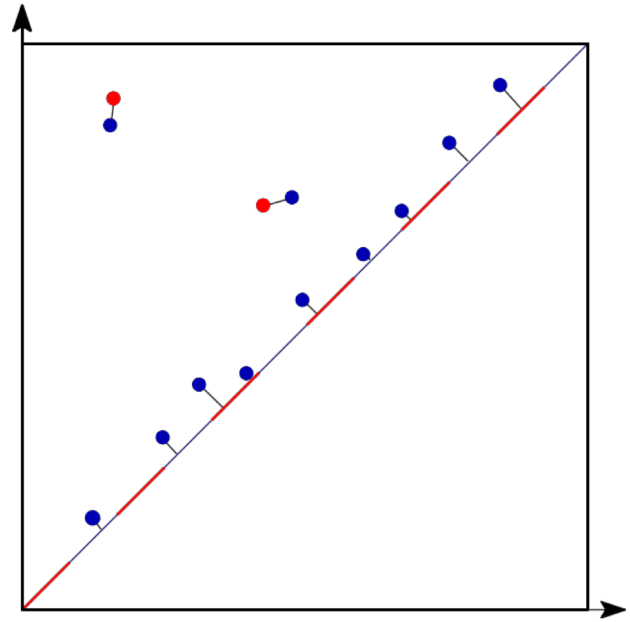




Bottleneck Distance

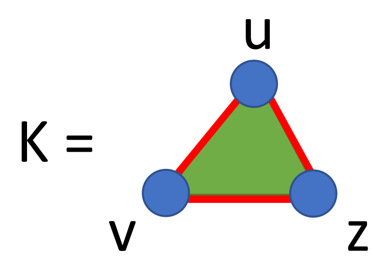
For persistence diagrams *U* and *V, B* the set of bijections from *U* to *V*. Then define the distance:

$$d_b(U,V) = \inf_{\gamma \in B} \sup_{x \in U} d_\infty(x, \gamma(x))$$

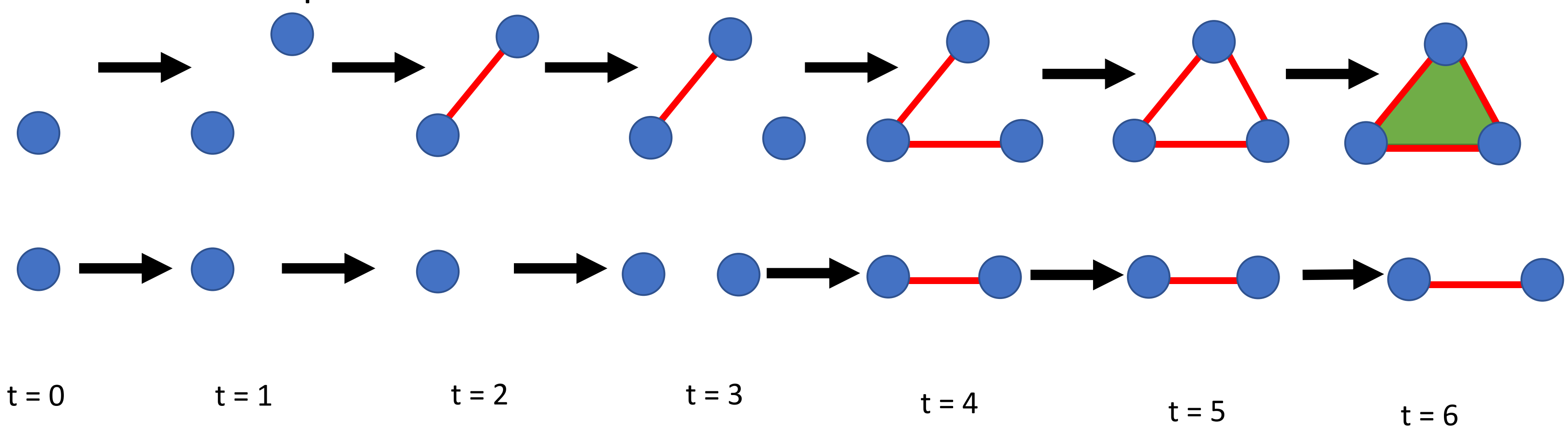


Edge Contraction

Viewed as a simplicial map $\xi_{\{u,v\}} \ : \ K \to \Delta$ where Δ is the maximal simplicial complex on the vertex set of K



Induces a new filtration $\xi_{\{u,v\}}(K_0) \subset \cdots \subset \xi_{\{u,v\}}(K_n)$ for the contracted complex

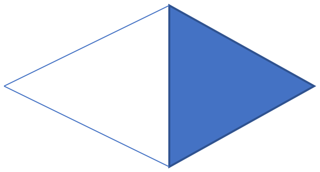


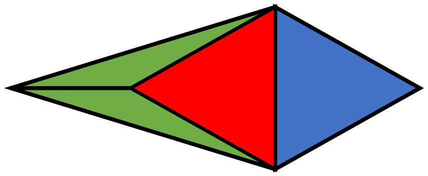
Edge Contraction

Define:

$$Cl(\sigma) = \{\tau \mid \tau < \sigma\}$$
$$St(\sigma) = \{\tau \mid \tau > \sigma\}$$
$$Lk(\sigma) = Cl(St(\sigma)) \setminus St(Cl(\sigma))$$

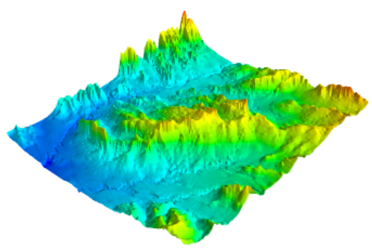
Link Condition (Dey et. al.) : $Lk(u) \cap Lk(v) = Lk(\{u, v\})$

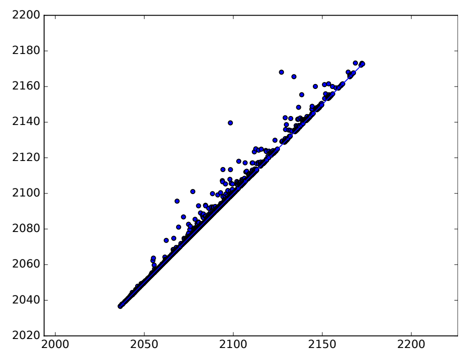


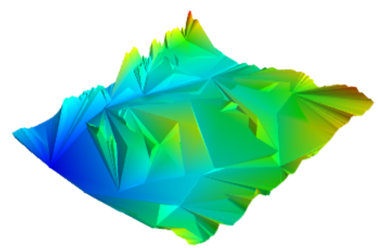


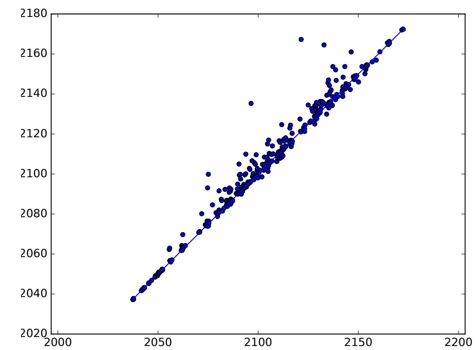
Goal

Establish a contraction operator that bounds the perturbation between the *p*-dimensional persistence diagrams









Modules

Define $K'_a := \xi_{u,v}(K_a)$

$$K_{i_1} \subseteq K_{i_2} \subseteq \ldots \subseteq K_{i_n}$$

$$H_k(K_{i_1}) \xrightarrow{f_{i_1,i_2}} H_k(K_{i_2}) \xrightarrow{f_{i_2,i_3}} \ldots \xrightarrow{f_{i_{n-1},i_n}} H_k(K_{i_n})$$

$$K'_{i_1} \subseteq K'_{i_2} \subseteq \ldots \subseteq K'_{i_n}$$

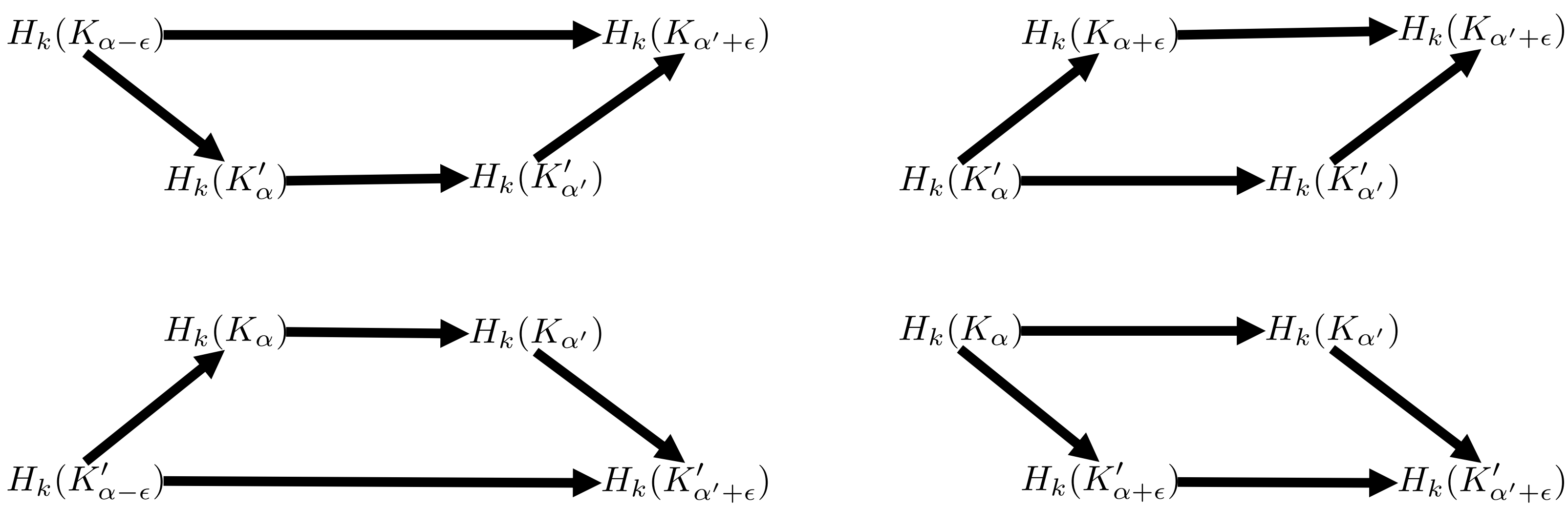
$$H_k(K'_{i_1}) \xrightarrow{f'_{i_1,i_2}} H_k(K'_{i_2}) \xrightarrow{f'_{i_2,i_3}} \ldots \xrightarrow{f'_{i_{n-1},i_n}} H_k(K'_{i_n})$$

ϵ-Interleavings

$$M \ : \ H_k(K_{i_1}) \xrightarrow{f_{i_1,i_2}} H_k(K_{i_2}) \xrightarrow{f_{i_2,i_3}} \ldots \xrightarrow{f_{i_{n-1},i_n}} H_k(K_{i_n})$$

$$M' \ : \ H_k(K'_{i_1}) \xrightarrow{f'_{i_1,i_2}} H_k(K'_{i_2}) \xrightarrow{f'_{i_2,i_3}} \ldots \xrightarrow{f'_{i_{n-1},i_n}} H_k(K'_{i_n})$$

Definition: Modules M and M' are ϵ-interleaved if there exist families of homomorphisms such that the following diagrams commute for all $\alpha < \alpha'$.



Theorem (Chazal et. al.): Let U be the persistence diagram corresponding to M, V corresponding to M'. If M and M' are ϵ-interleaved, then $d_b(U, V) \leq \epsilon$.

Defining Homomorphisms

Downward maps $\xi^*_{a,a+\epsilon} \; : \; H_k(K_a) \to H_k(K'_{a+\epsilon})$.

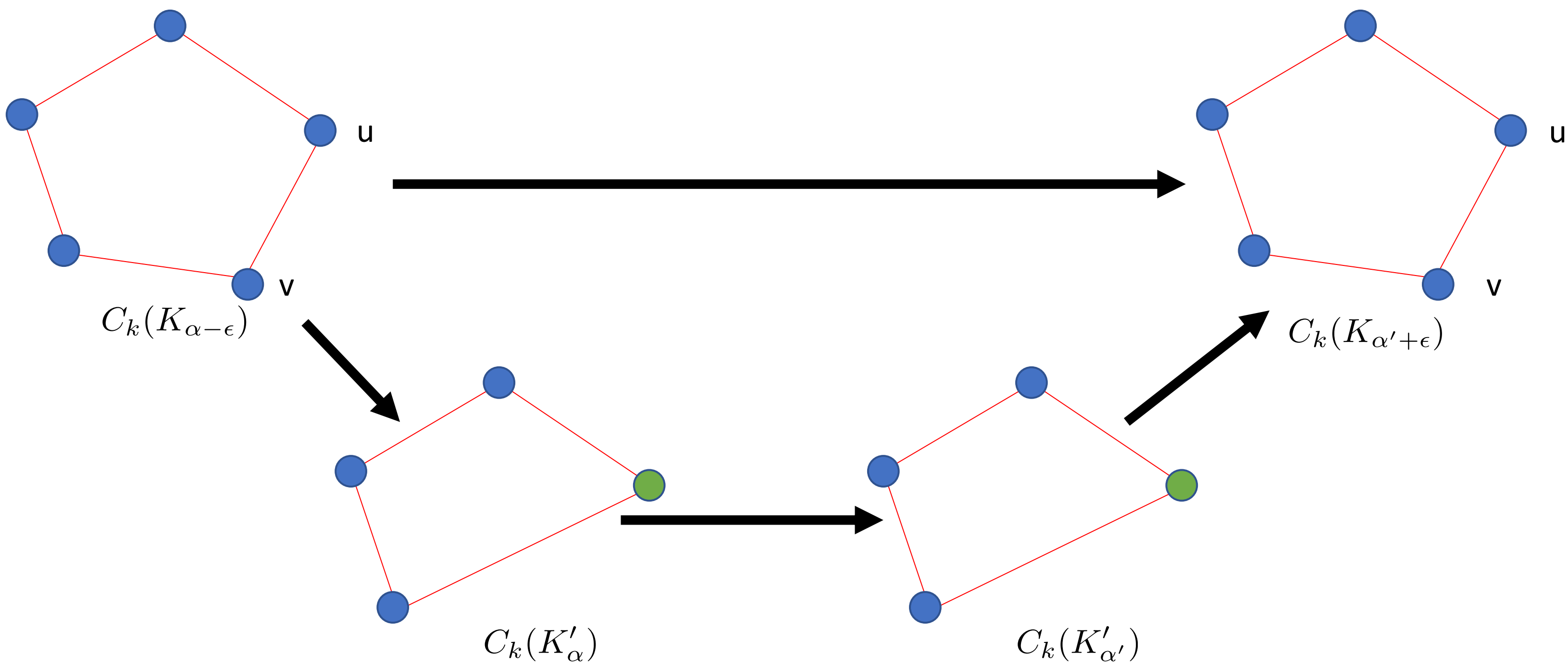
Edge contraction induces chain maps $\xi_{a,\#} \; : \; C_k(K_\alpha) \to C_k(K'_\alpha)$

Theorem $\xi_{\alpha,\#}$ brings cycles to cycles and boundaries to boundaries.

Homology maps: $\xi^*_a \; : \; H_k(K_\alpha) \to H_k(K'_\alpha)$

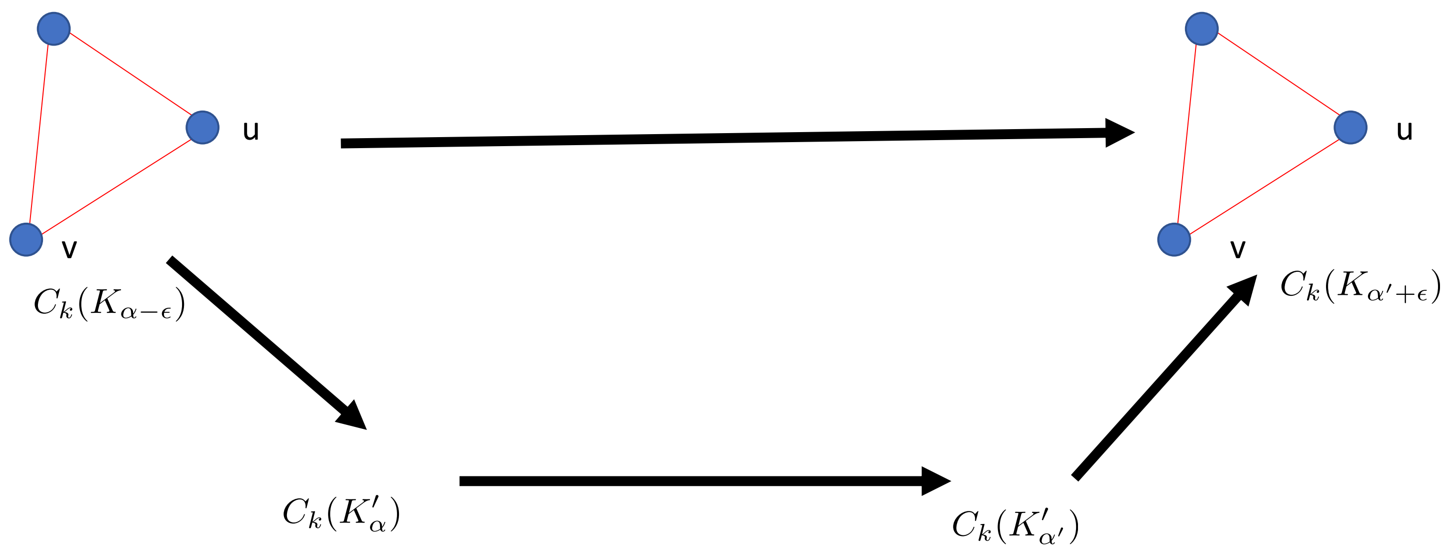
Hence, $\xi^*_{\alpha,\alpha+\epsilon} := f'_{\alpha,\alpha'} \circ \xi^*_\alpha$

Intuition

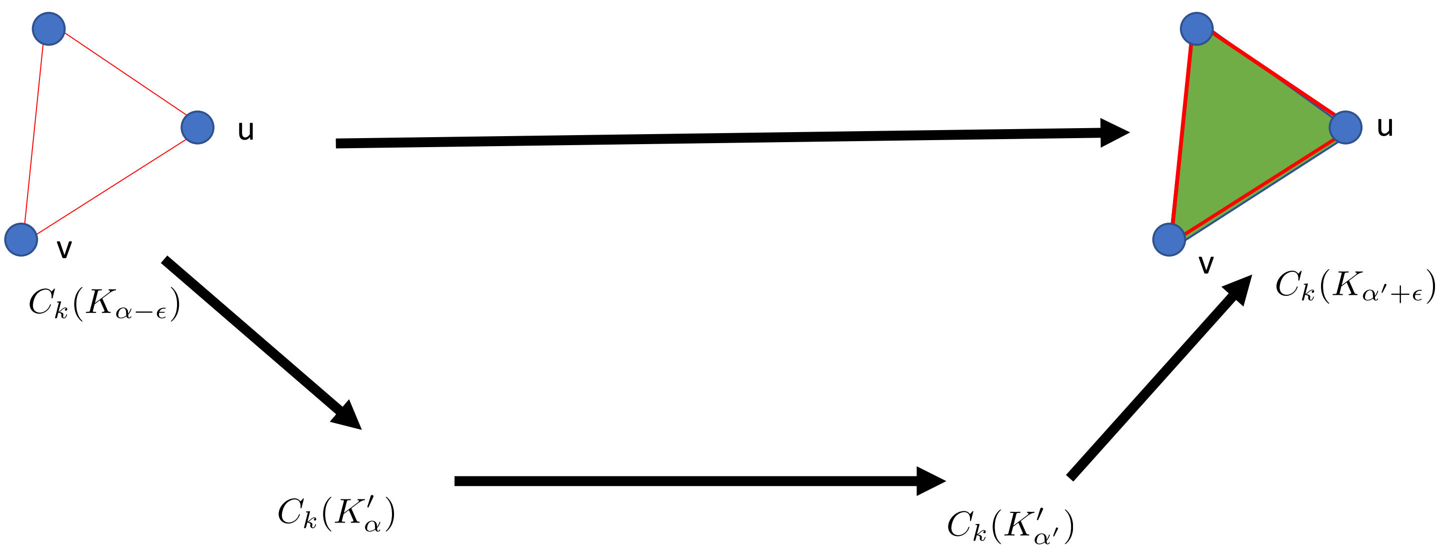


u
v
$C_k(K_{\alpha-\epsilon})$
u
v
$C_k(K_{\alpha'+\epsilon})$
$C_k(K'_{\alpha})$
$C_k(K'_{\alpha'})$

Intuition

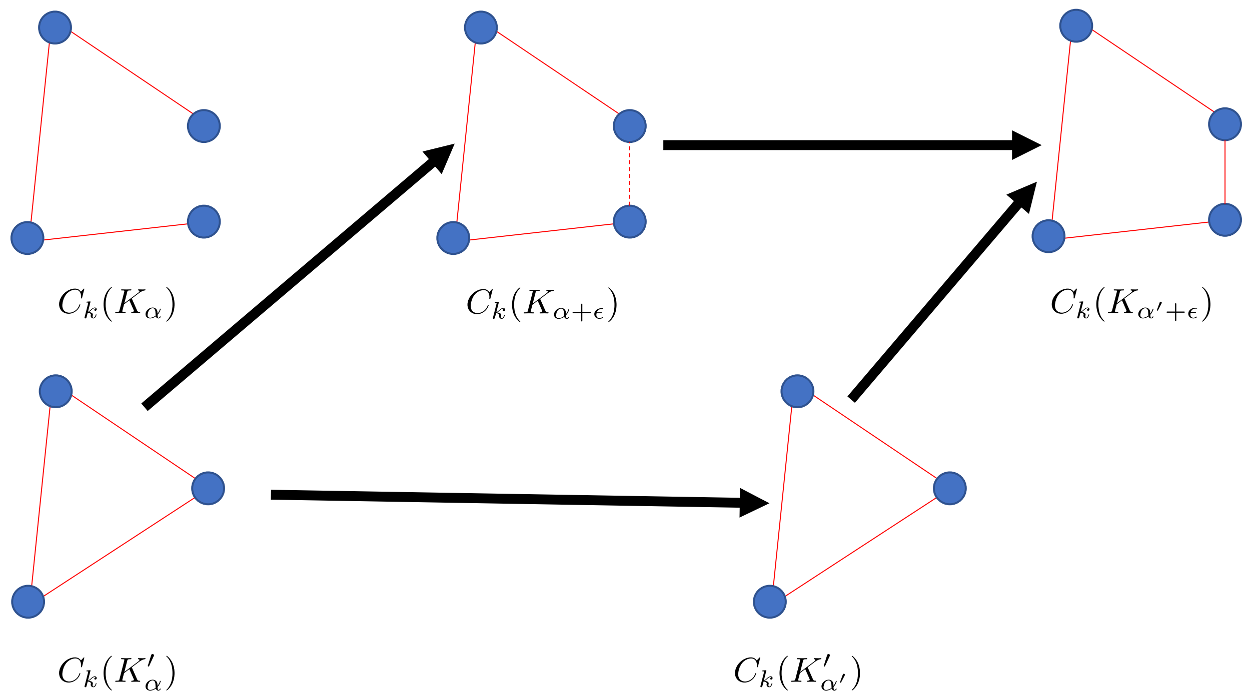

u
v
$C_k(K_{\alpha-\epsilon})$
u
v
$C_k(K_{\alpha'+\epsilon})$
$C_k(K'_{\alpha})$
$C_k(K'_{\alpha'})$

Intuition



u
v
$C_k(K_{\alpha-\epsilon})$
u
v
$C_k(K_{\alpha'+\epsilon})$
$C_k(K'_{\alpha})$
$C_k(K'_{\alpha'})$

Intuition



$C_k(K_\alpha)$
$C_k(K_{\alpha+\epsilon})$
$C_k(K_{\alpha'+\epsilon})$
$C_k(K'_\alpha)$
$C_k(K'_{\alpha'})$

Conditions

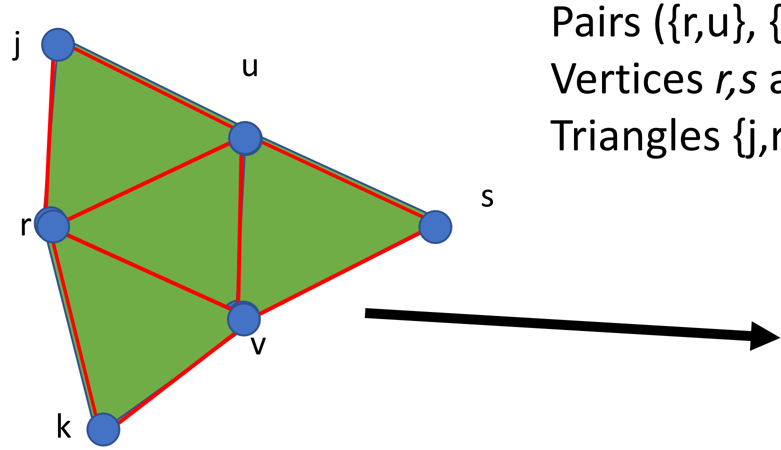
Contracting {u,v}

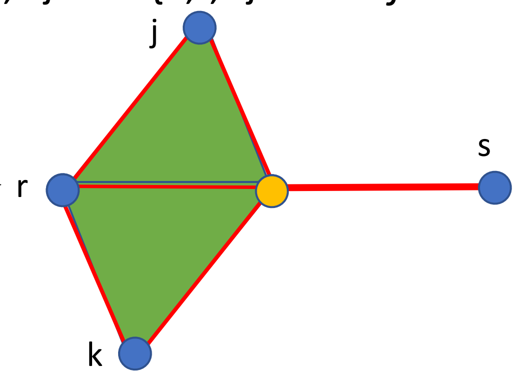
Triangles {r,u,v}, {s,u,v}, and edge {u,v} are *vanishing*

Pairs ({r,u}, {r,v}), ({s,u},{s,v}), and ({u,v}) are *mirrored*

Vertices *r,s* are *bystanders*

Triangles {j,r,u} and {k,r,v} are *adjacent*





Edge e is (p,ϵ)-admissible if:

- For each pair of *p*-mirrors s_1, s_2 , where $h(s_1) < h(s_2)$ with shared vanishing cofacet t,$|h(s_2) - h(t)| < \epsilon$.
- If p>0, for each pair of *(p-1)*-mirrors s_1, s_2 , where $h(s_1) < h(s_2)$ with shared vanishing cofacet t,$|h(s_2) - h(t)| < \epsilon$.

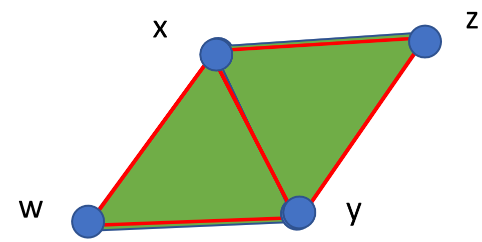
Upward Maps

Require {u,v} to be (p,ϵ)-admissible. Define $\psi_{\alpha,\alpha+\epsilon} : K'_\alpha \to C_k(K_{\alpha+\epsilon})$ which extends linearly to $\psi_{\alpha,\alpha+\epsilon,\#} : C_k(K'_\alpha) \to C_k(K_{\alpha+\epsilon})$

Gives homology maps $\psi^*_{\alpha,\alpha+\epsilon} : H_k(K'_\alpha) \to H_k(K_{\alpha+\epsilon})$

But how to define $\psi_{\alpha,\alpha+\epsilon} : K'_\alpha \to C_k(K_{\alpha+\epsilon})$?

Upward Maps



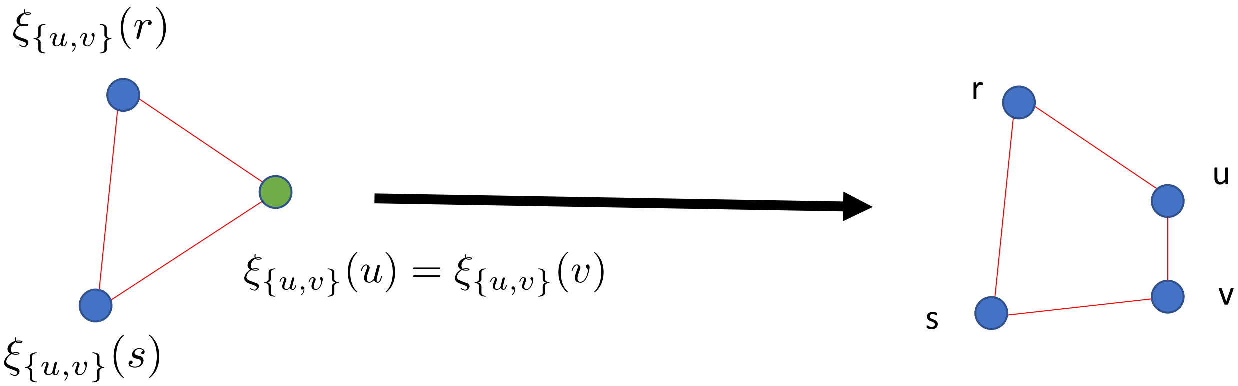
Triangles {w,x,y} and {x,y,z} are *incident*

If σ is nonlocal or a bystander, then $\psi_{\alpha,\alpha+\epsilon}(\sigma) = \sigma$.

If σ is the image of mirrors, $\sigma = \xi_{\{u,v\}}(\sigma_u) = \xi_{\{u,v\}}(\sigma_v)$, $\sigma_u \prec \sigma_v$, then $\psi_{\alpha,\alpha+\epsilon}(\sigma) = \sigma_u + \sum_I \sigma_i$ where σ_i are vanishing and incident to σ_u by a younger mirror.

If σ is adjacent, then $\psi_{\alpha,\alpha+\epsilon}(\sigma) = \xi^{-1}_{\{u,v\}}(\sigma) + \sum_I \sigma_i$ where σ_i are vanishing and incident to $\xi^{-1}_{\{u,v\}}(\sigma)$ by a younger mirror.

Upward Maps



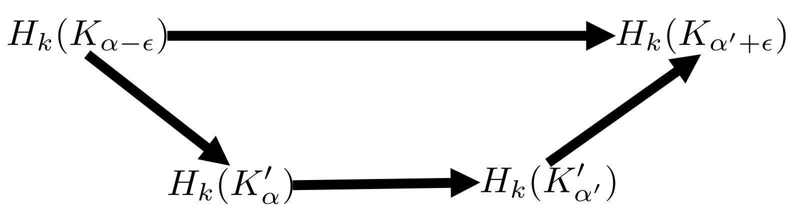
If *u* is younger than *v*, then $\psi_{\alpha,\alpha+\epsilon}(\{\xi_{\{u,v\}}(r), \xi_{\{u,v\}}(u)\}) = \{r, u\} + \{u, v\}$

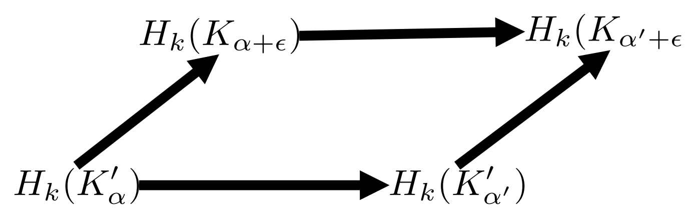
Else, $\psi_{\alpha,\alpha+\epsilon}(\{\xi_{\{u,v\}}(s), \xi_{\{u,v\}}(u)\}) = \{s, v\} + \{u, v\}$

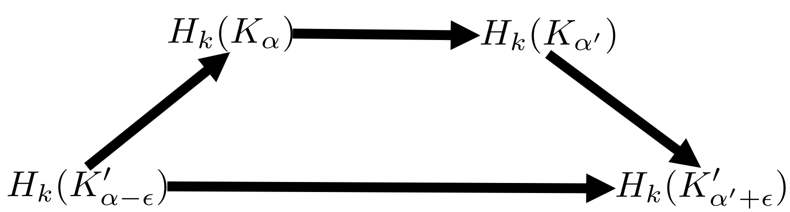
Theorem $\psi_{\alpha,\alpha+\epsilon}$ brings cycles to cycles and boundaries to boundaries.

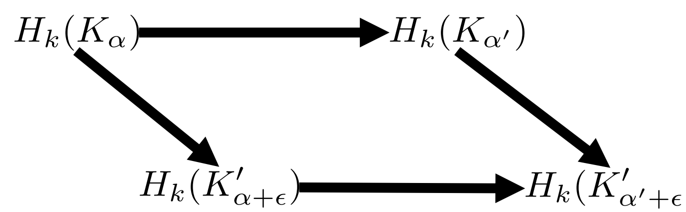
Commutation

Do the maps $\xi^*_{\alpha,\alpha+\epsilon}$, $\psi^*_{\alpha,\alpha+\epsilon}$, $f'_{\alpha,\alpha+\epsilon}$, $f_{\alpha,\alpha+\epsilon}$ commute?

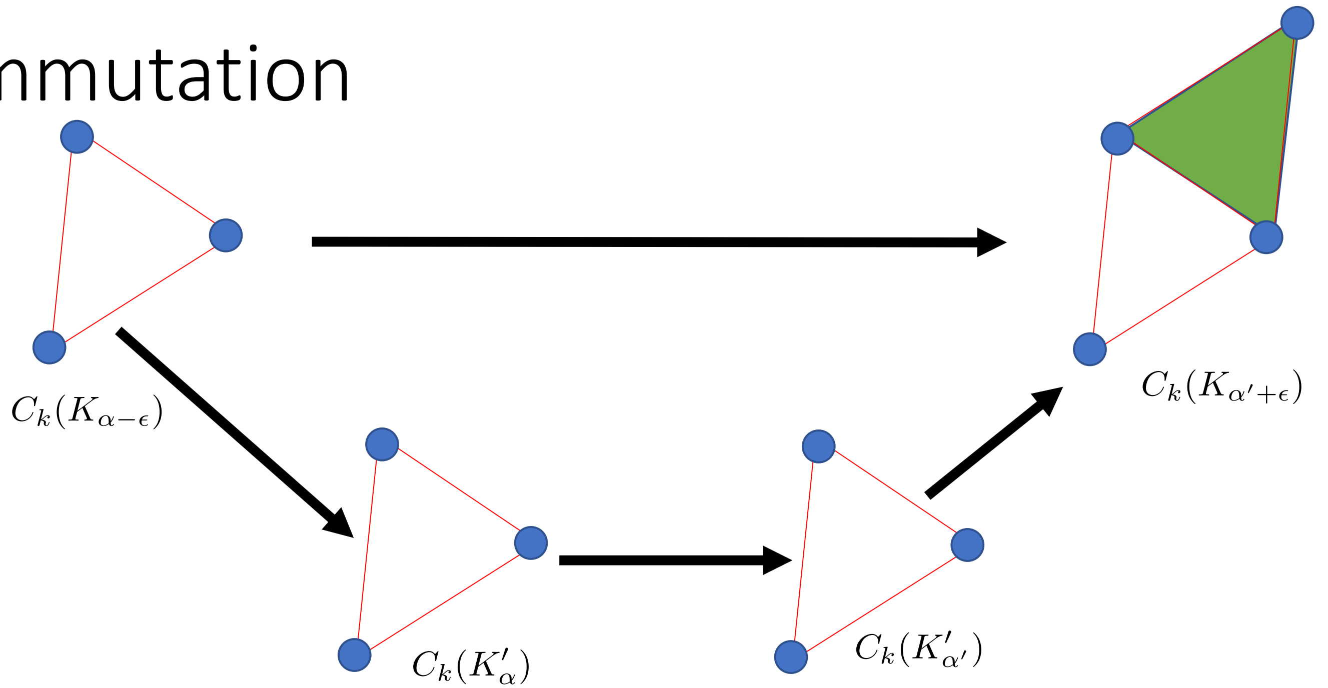








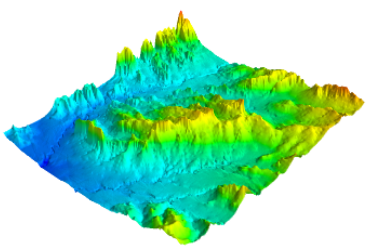
Commutation

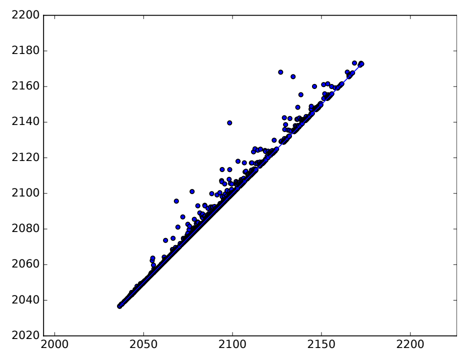


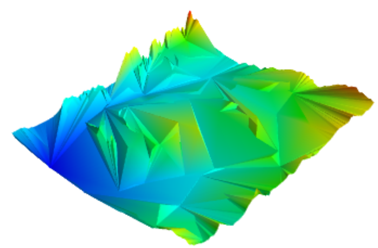
Theorem Let {u,v} be (p,ϵ)-admissible. For filtered simplicial complex *K*, let *U* be the *p*-dimensional persistence diagram corresponding to $K_{i_1} \subseteq K_{i_2} \subseteq \ldots \subseteq K_{i_n}$ and *V* the *p*-dimensional persistence diagram corresponding to $K'_{i_1} \subseteq K'_{i_2} \subseteq \ldots \subseteq K'_{i_n}$. Then $d_b(U,V) \leq \epsilon$.

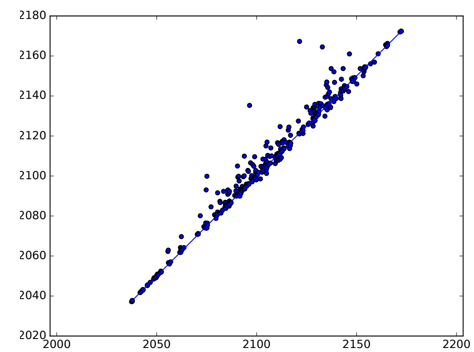
Multiple Contractions

Can we contract multiple (p, ϵ)-edges such that $d_b(U,V) \leq \epsilon$?

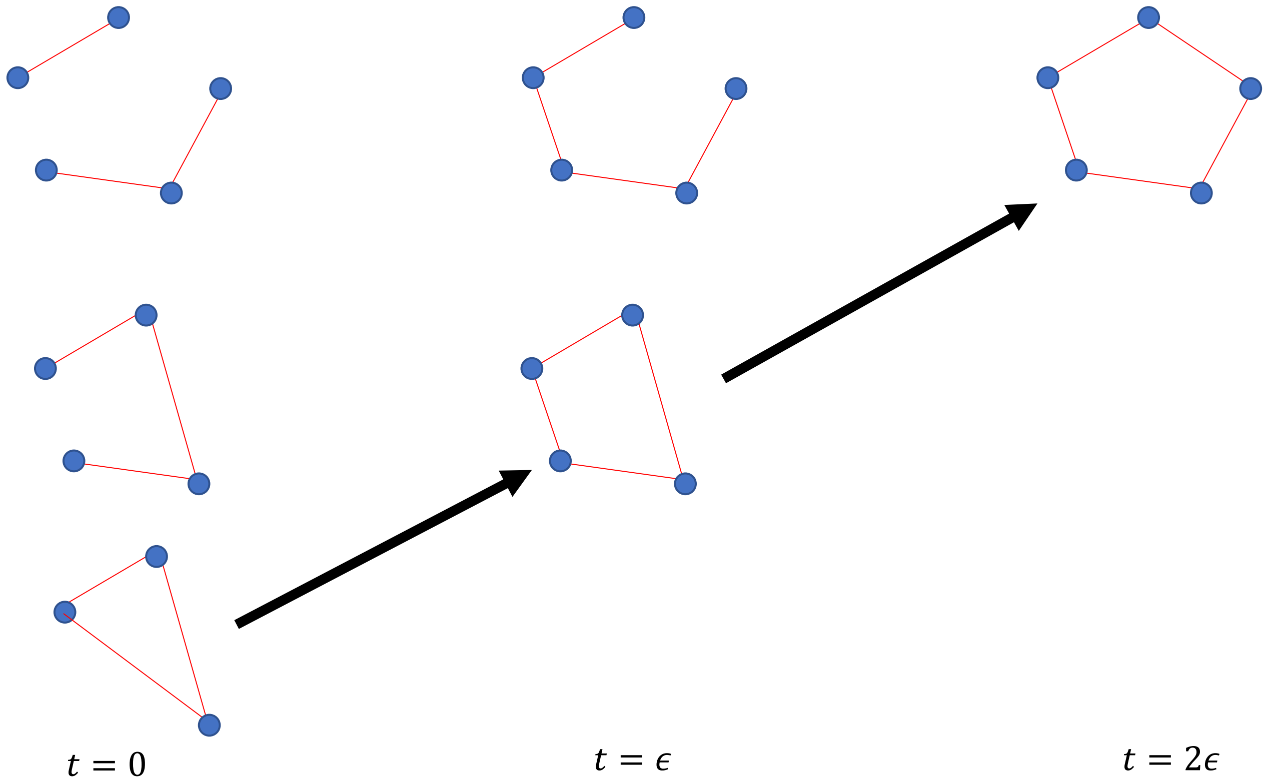








Multiple Contractions



Multiple Contractions

For each (p, ϵ)-edge *{u,v}*, define $W(\{u, v\}) = [a, b]$

- *a* the oldest (*p-1*) – simplex
- *b* the oldest *(p+1)* – simplex

Theorem For a collection of (p, ϵ)-edges e_1, e_2, …, e_m, if $W(e_1)$, $W(e_2)$,…,$W(e_m)$ are disjoint, then the bottleneck distance between the initial and final *p*-dimensional persistence diagram is $\leq \epsilon$**.**

Experiments

Contracting (1,5.0) edges.
Image corresponds to Los Alamos.

Dataset	# Cont.	% Red.	# Sets	Bottleneck	Bound
Columbus	452,914	99.6	6,123	2.097	30,615
Los Alamos	452,710	99.6	7,350	6.540	36,750
Minneapolis	1,819,786	99.8	66,981	4.832	334,905
Aspen	1,818,593	99.7	5,664	8.570	28,320
Filigree	466,395	90.7	90,009	9.051	450,045
Eros	476,573	99.99	158,426	4.296	792,130
Statue	2,470	98.8	945	0.074	4,725