Tracking Dynamical Features via Continuation and Persistence

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PURDUE UNIVERSITY

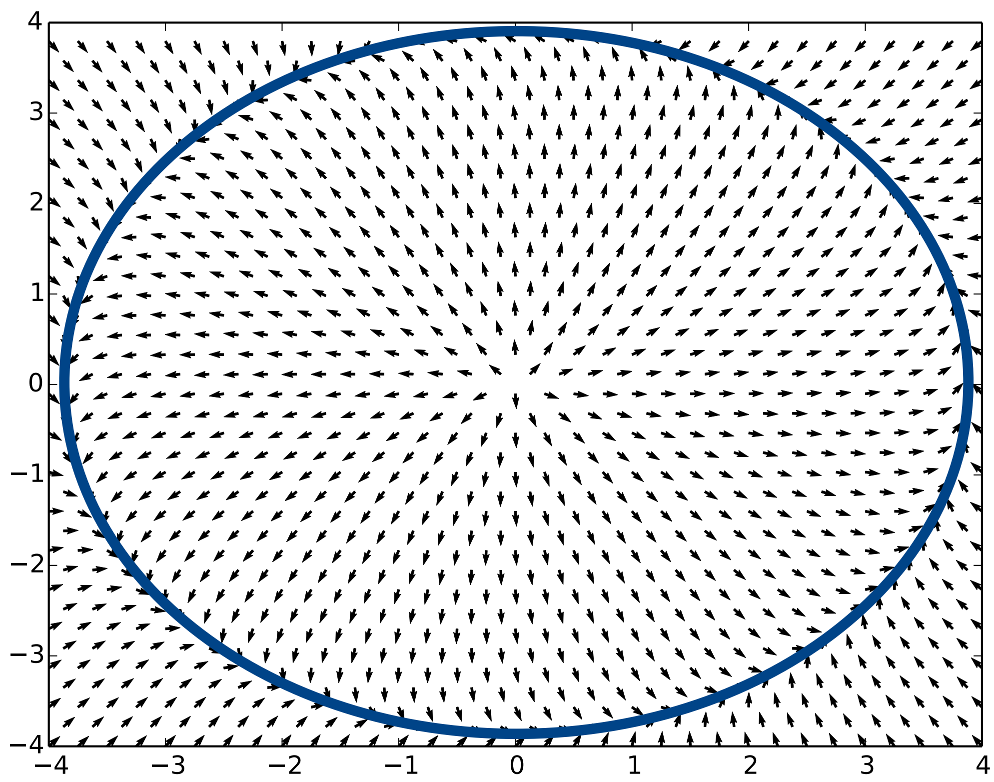
JAGIELLONIAN UNIVERSITY IN KRAKOW

Motivating Example: Hopf Bifurcation

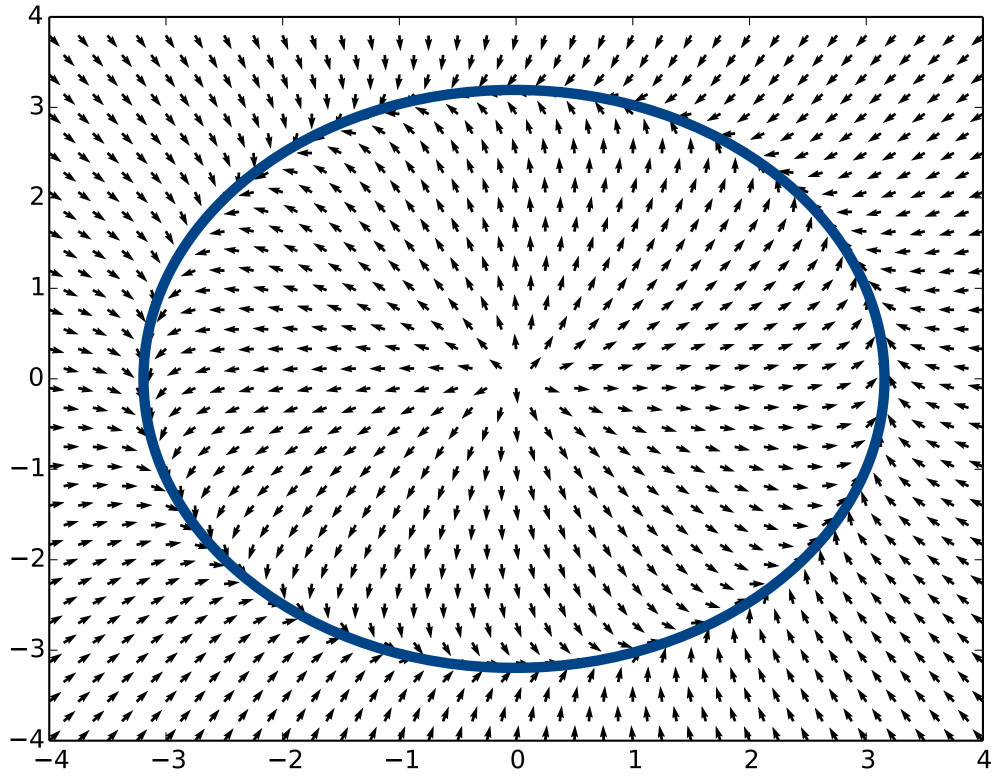
$$x' = -y + x(\lambda - x^2 - y^2)$$

$$y' = x + y(\lambda - x^2 - y^2)$$

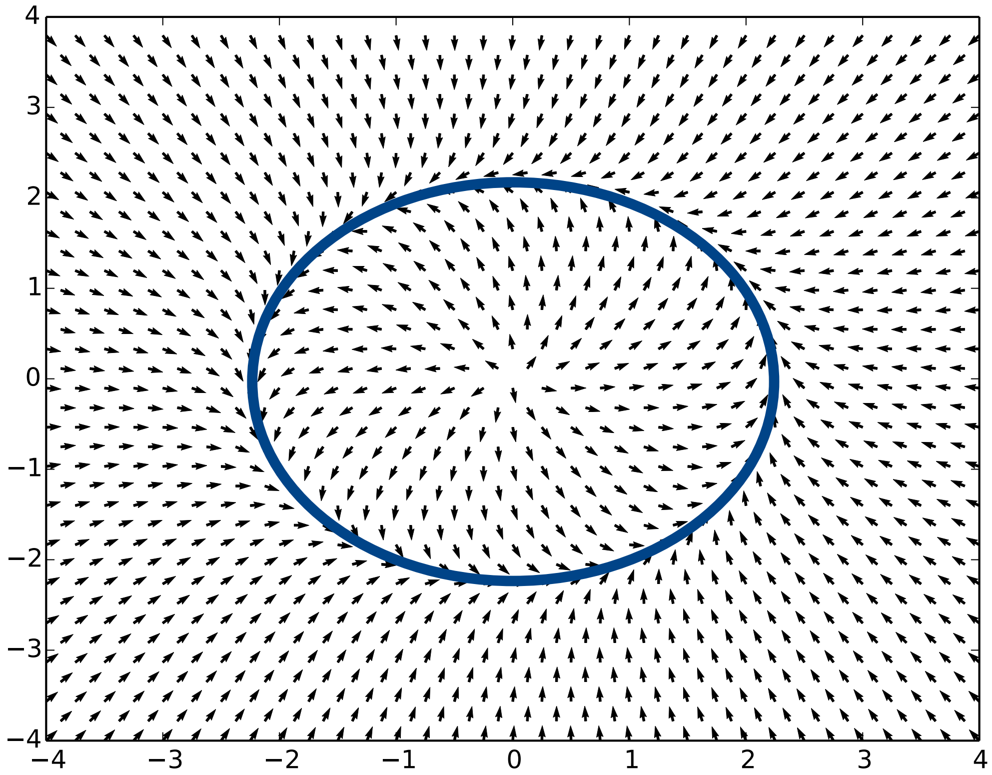
Motivating Example: Hopf Bifurcation



Motivating Example: Hopf Bifurcation



Motivating Example: Hopf Bifurcation



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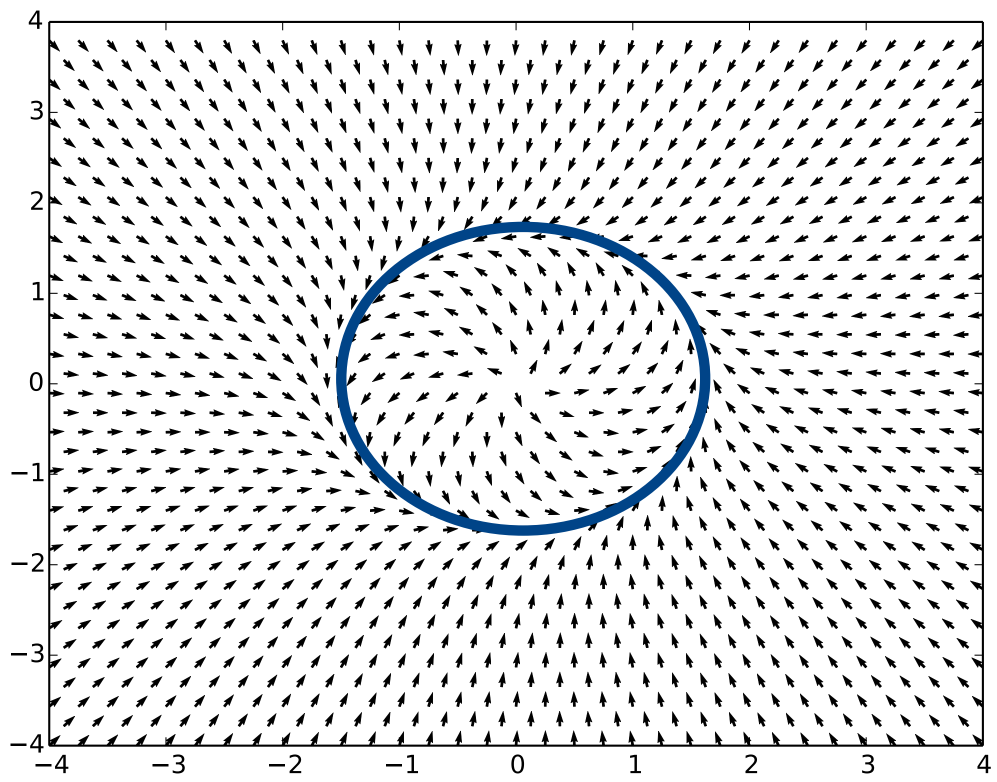
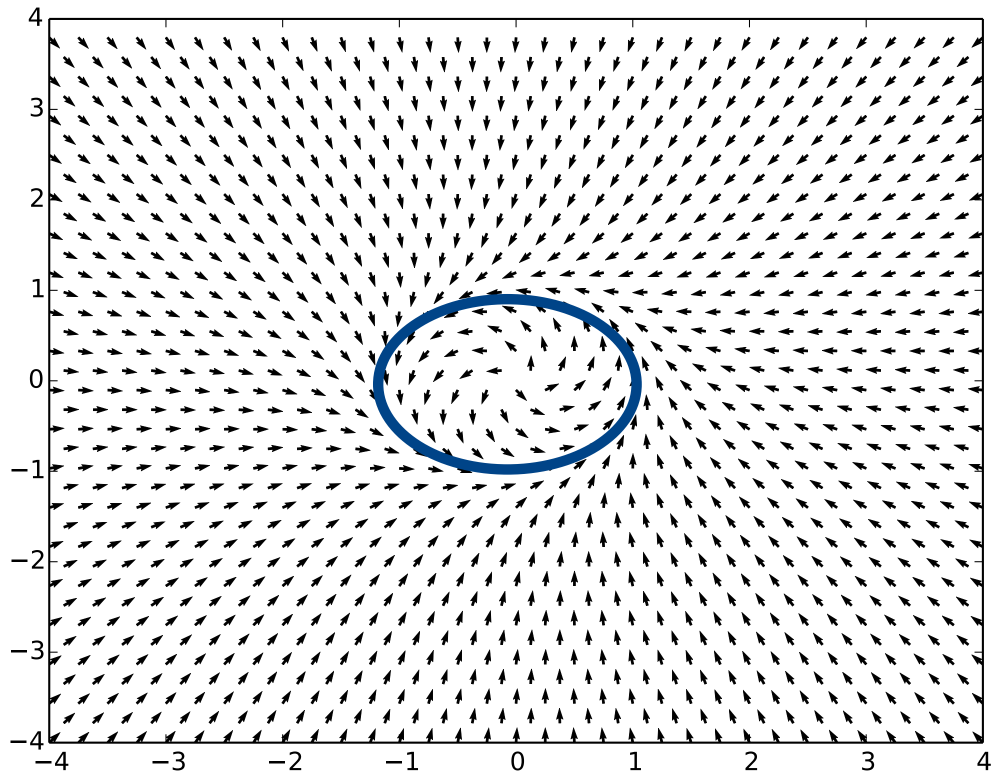
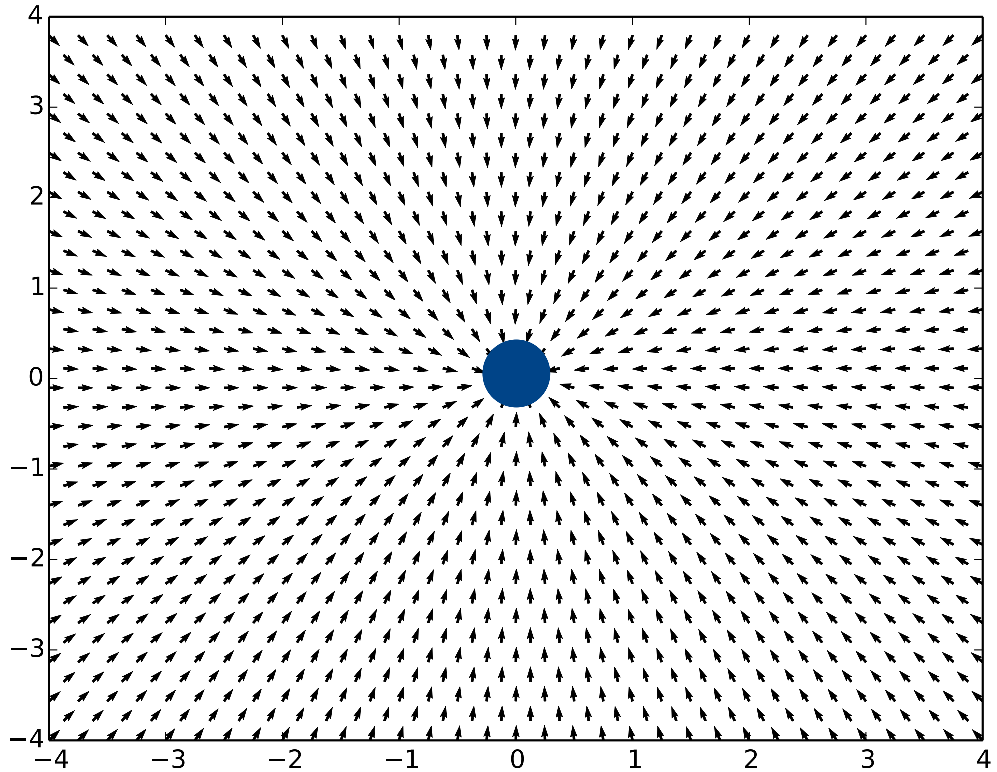


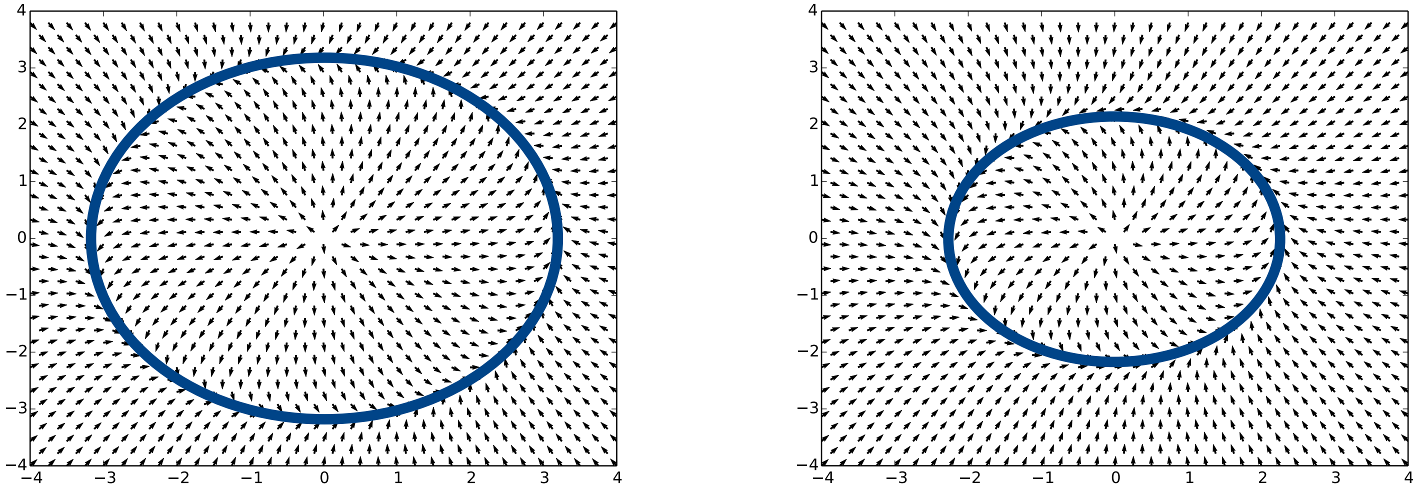
Motivating Example: Hopf Bifurcation



Motivating Example: Hopf Bifurcation

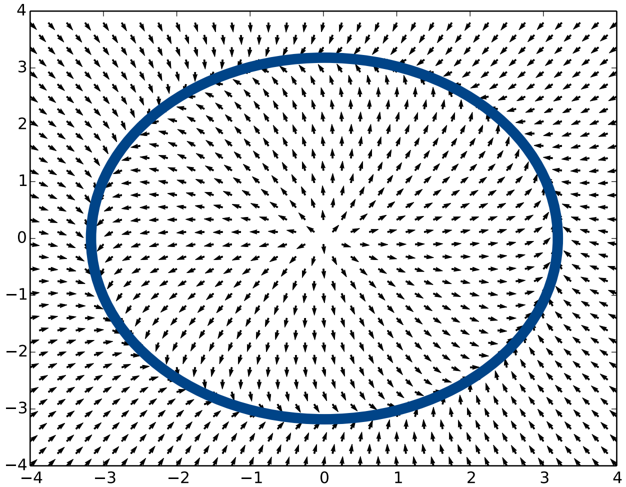


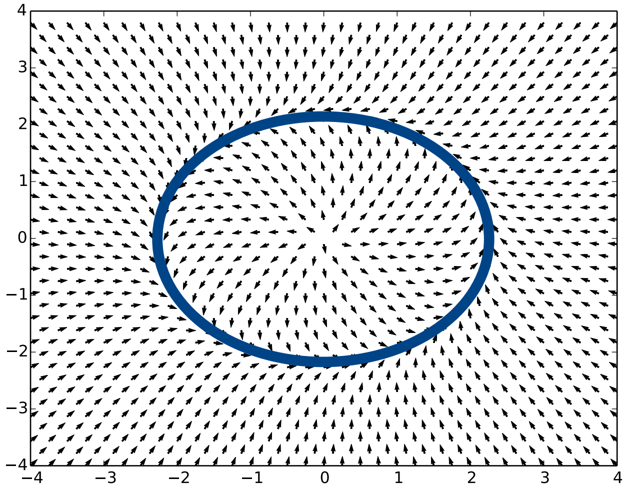
“The Same” Invariant Set



Theorem [Conley, Eaton, ’71]: If f is a flow and N is an isolating neighborhood for f, and if f’ is a sufficiently small perturbation of f, then N is an isolating neighborhood for f’.

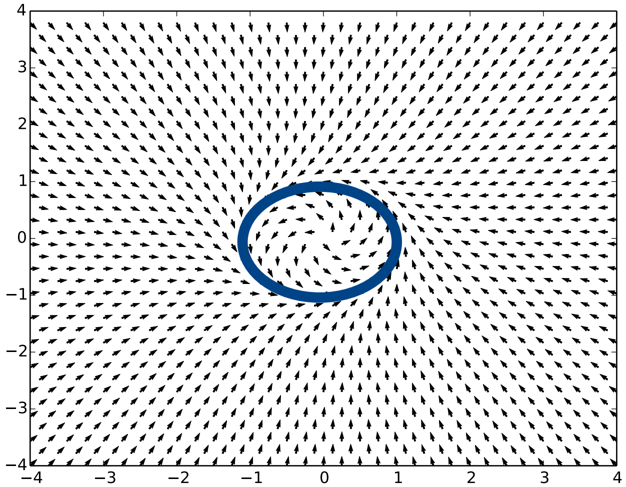
“The Same” Invariant Set

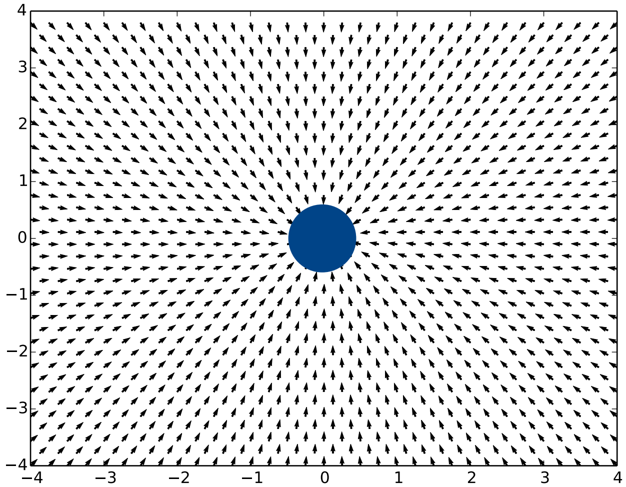




The two invariant sets are related by a *continuation.*

Different Invariant Set





The Problem

In the combinatorial setting, can we…

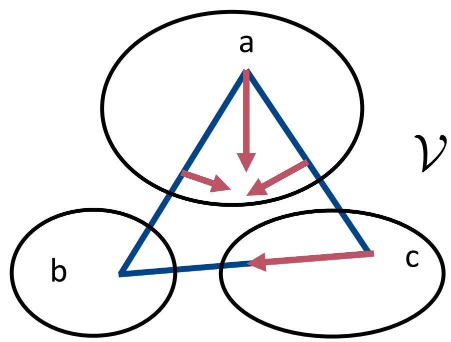
1. Track the presence of “the same” invariant set across multiple dynamical systems?
2. Use persistence to continue tracking in the case of a bifurcation (or a reverse-bifurcation)?

Multivectors

Let K denote a simplicial complex and $\leq$ denote the face relation.

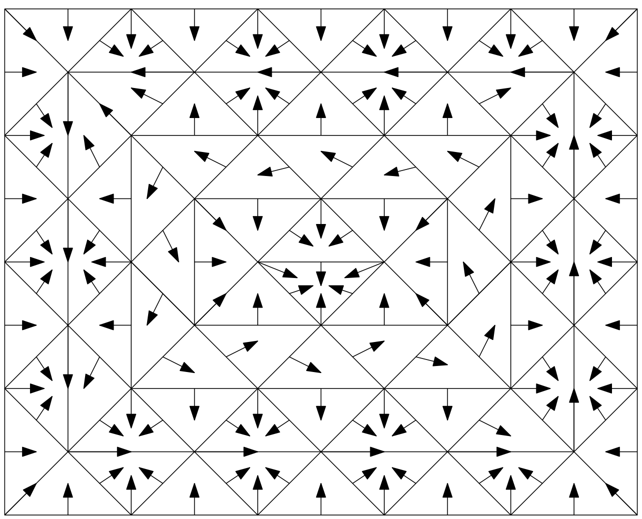
Definition: A multivector V is a convex subset of K with respect to $\leq$.

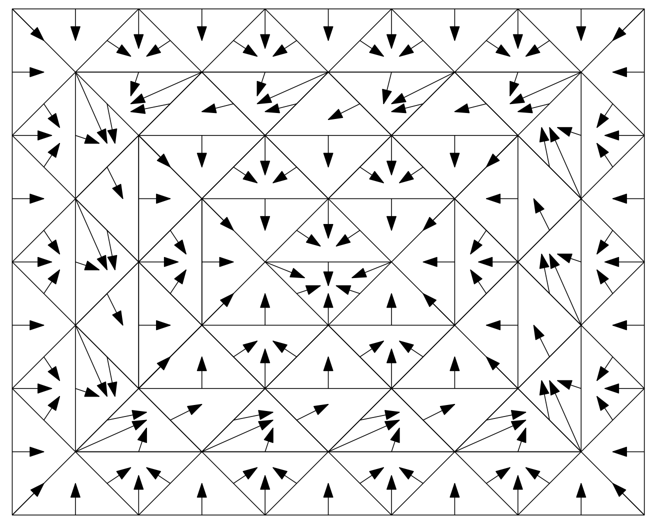
Definition: A multivector field $\mathcal{V}$ is a partition of K into multivectors.



$$\mathcal{V} = \{\{b\}, \{c, bc\}, \{a, ab, ac, abc\}\}$$

Multivector Fields





Multivector Fields as a Dynamical System

Let $\sigma \in K$. Then $\mathrm{cl}(\sigma) = \{\tau \in K \mid \tau \leq \sigma\}$.

$[\sigma]_{\mathcal{V}}$ denotes the vector in $\mathcal{V}$ containing σ

Dynamics generator $F_{\mathcal{V}} \; : \; K \multimap K$ defined as:

$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$$

Multivector Fields as a Dynamical System

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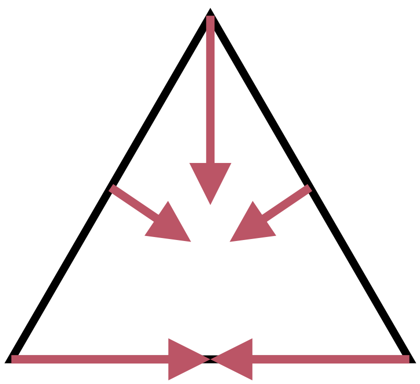


Multivector field

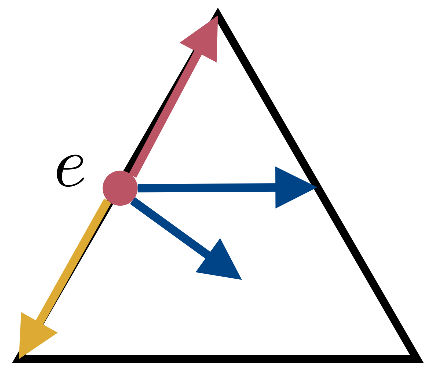


Image of marked edge e

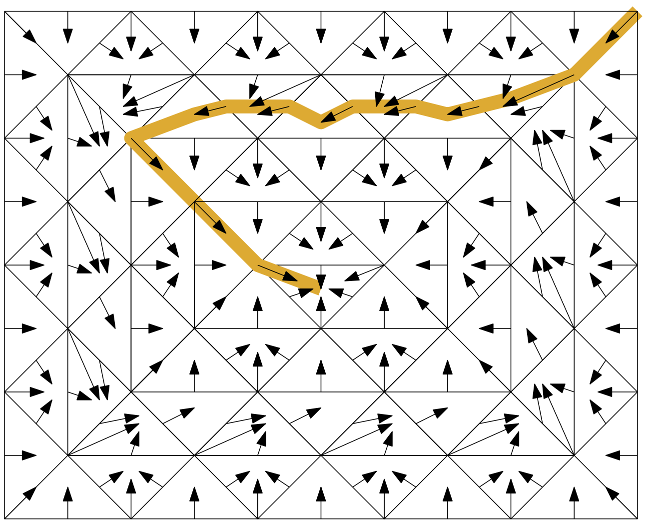
Blue simplices are in $[e]_{\mathcal{V}} \setminus \mathsf{cl}(e)$

Yellow simplices are in $\mathsf{cl}(e) \setminus [e]_{\mathcal{V}}$

Red simplices are in $[e]_{\mathcal{V}} \cap \mathsf{cl}(e)$

Paths

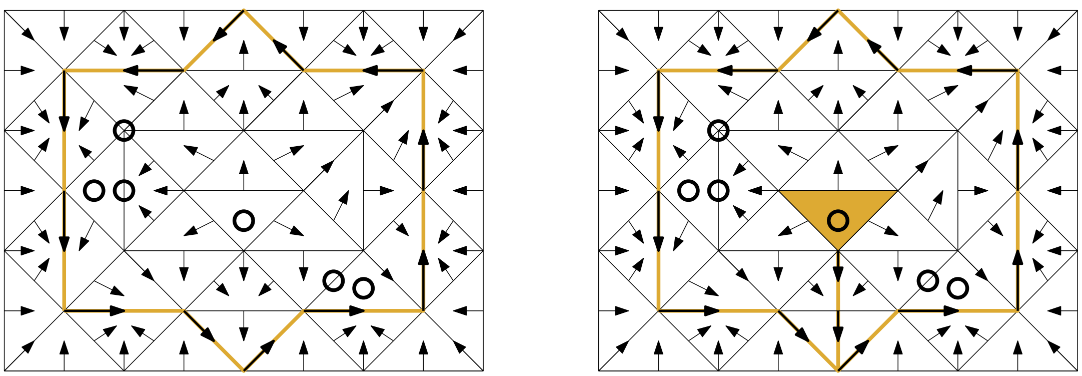
Definition: A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



Solutions

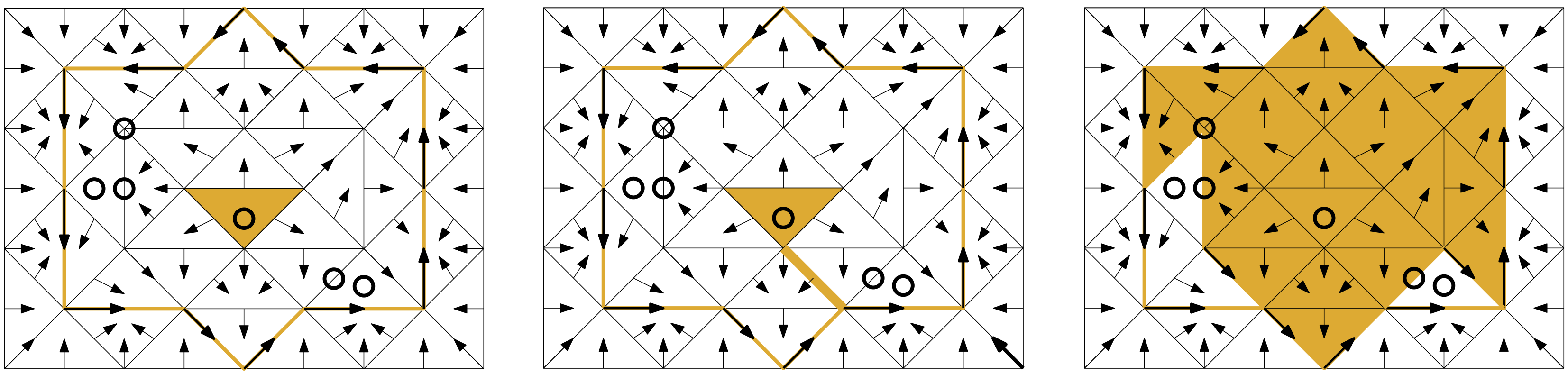
Definition: A solution is a bi-infinite sequence of simplices

$\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



Invariant Sets

Definition: Let $A \subseteq K$. The invariant part of A, denoted $\text{Inv}(A)$, is the set of simplices in A which appear in an essential solution in A.

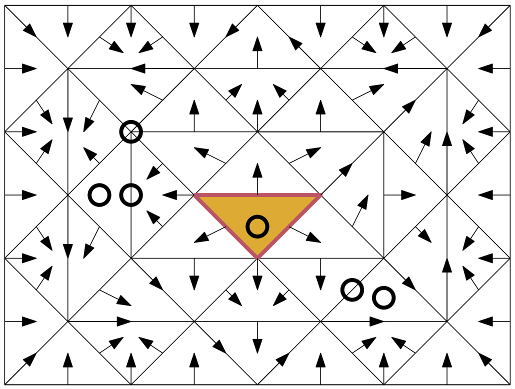


If $A = \text{Inv}(A)$, then A is an invariant set.

Index Pairs

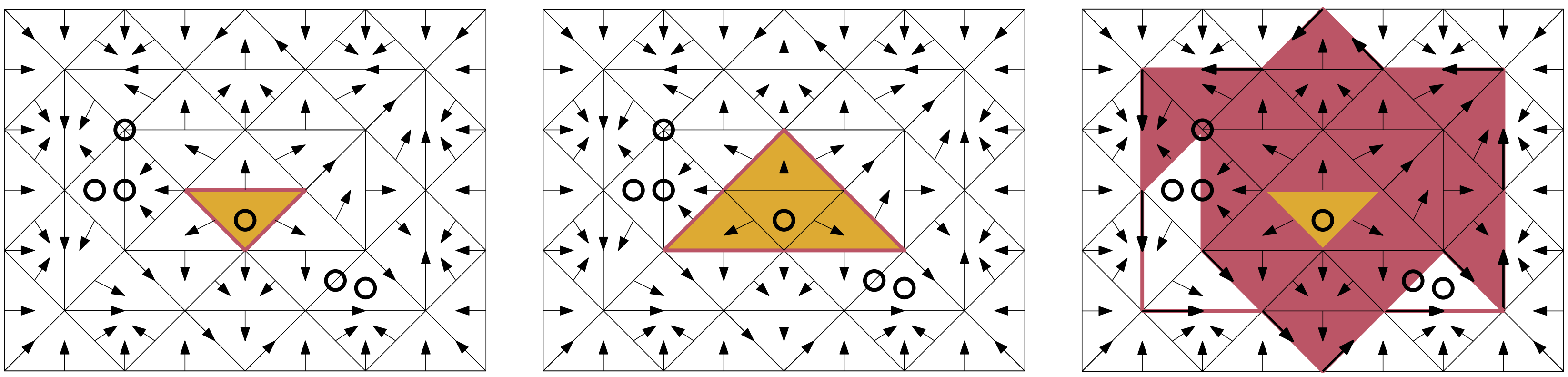
Definition: Let A be an isolated invariant set, and E and P closed sets such that $E \subseteq P$. If:

1. $F_{\mathcal{V}}(E) \cap P \subset E$,
2. $F_{\mathcal{V}}(P \setminus E) \subseteq P$, and
3. $A = \mathrm{Inv}(P \setminus E)$



Then (P, E) is an index pair for A.

Index Pairs are Not Unique

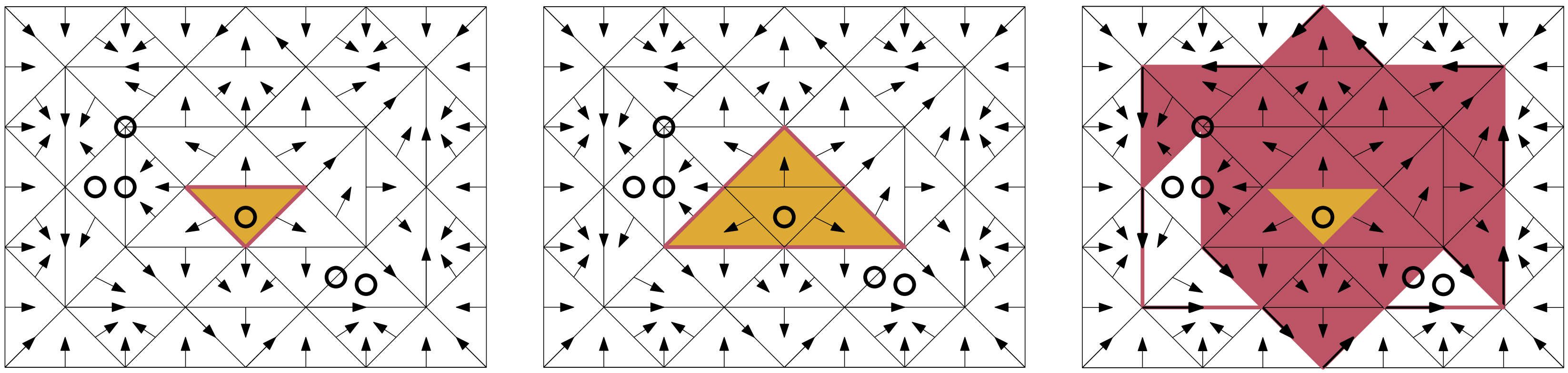


Conley Index

Definition: Let (P, E) be an index pair for A. Then the k-dimensional Conley Index is given by $H_k(P, E)$.

Theorem [LKMW 2019]: The k-dimensional Conley Index for A is well defined.

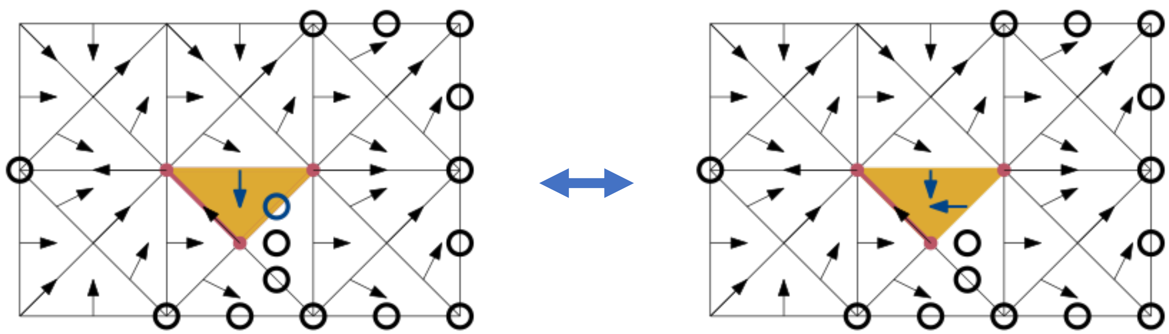
Conley Indices



$$H_2(R \cup Y, R) = \mathbb{Z}_2$$

Combinatorial Continuation

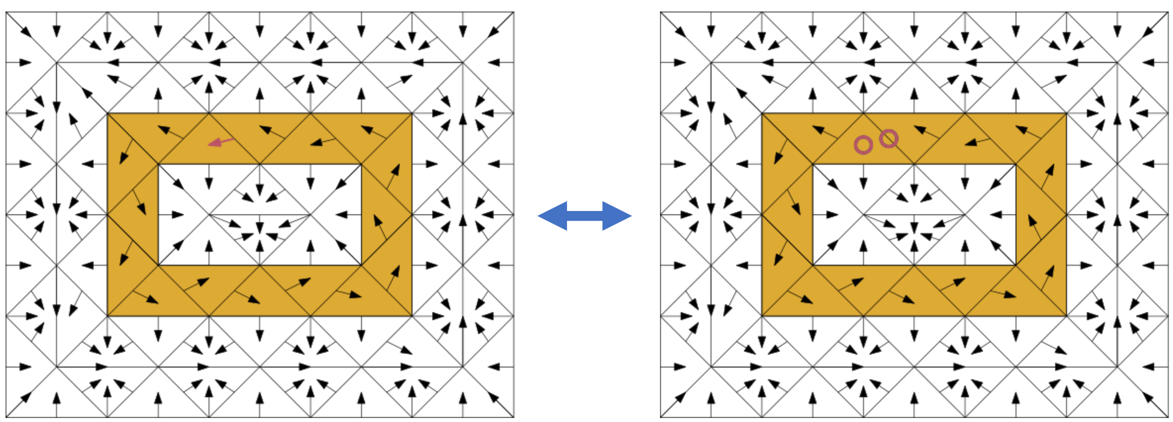
Definition: A sequence of isolated invariant sets is a *continuation* if there is a common index pair for each consecutive pair in the sequence.



Small Perturbations

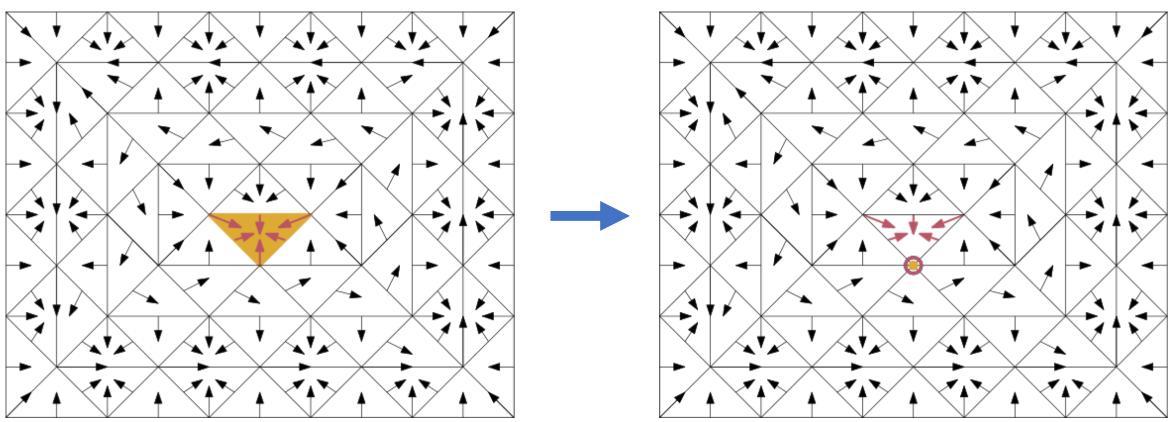
Definition: Let $\mathcal{V}$ denote a multivector field. The multivector field $\mathcal{V}'$ is an *atomic refinement* of $\mathcal{V}$ if $|\mathcal{V} \setminus \mathcal{V}'| = 1$ and $|\mathcal{V}' \setminus \mathcal{V}| = 2$. Symmetrically, $\mathcal{V}$ is an *atomic coarsening* of $\mathcal{V}'$.

Theorem[DLMS22]: Every pair of multivector fields can be transformed into each other by a sequence of atomic rearrangements.



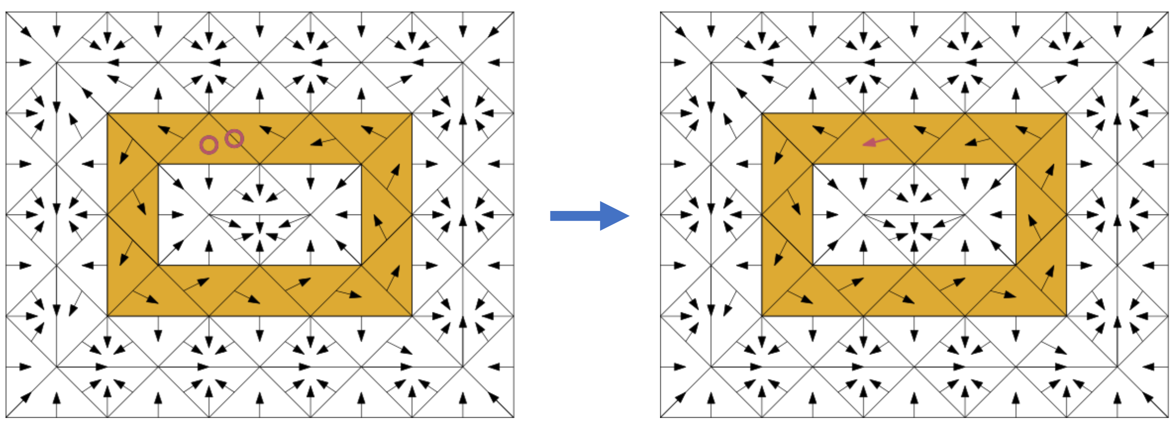
Tracking with Continuation: Case 1

<u>Theorem[DLMS22]:</u> Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic refinement of $\mathcal{V}$. Then $\mathsf{Inv}_{\mathcal{V}'}(\mathsf{S})$ is an invariant set and $(\mathsf{cl}(\mathsf{S}), \mathsf{mo}(\mathsf{S}))$ is an index pair for both.



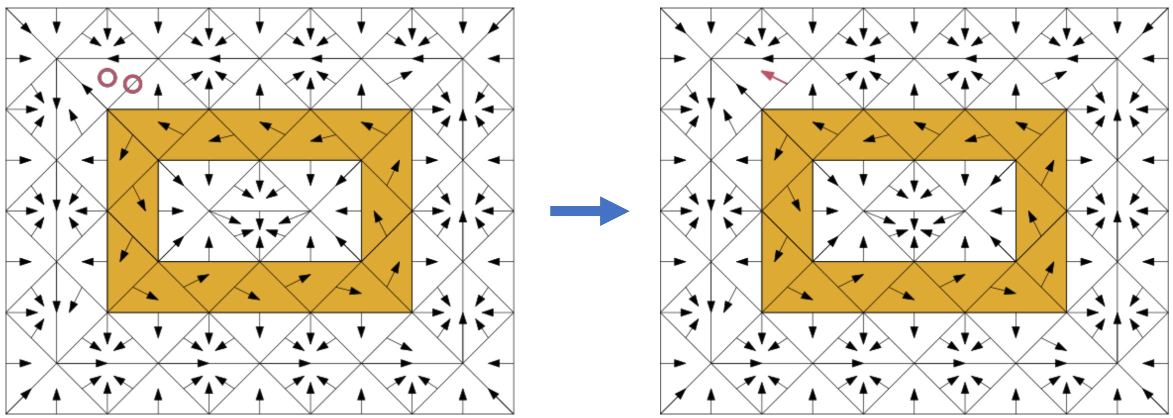
Tracking with Continuation: Case 2

Theorem[DLMS22]: Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique bifurcating multivector is contained in S. Then $\mathrm{Inv}_{\mathcal{V}'}(\mathrm{S})$ is an invariant set and $(\mathrm{cl}(\mathrm{S}), \mathrm{mo}(\mathrm{S}))$ is an index pair for both.



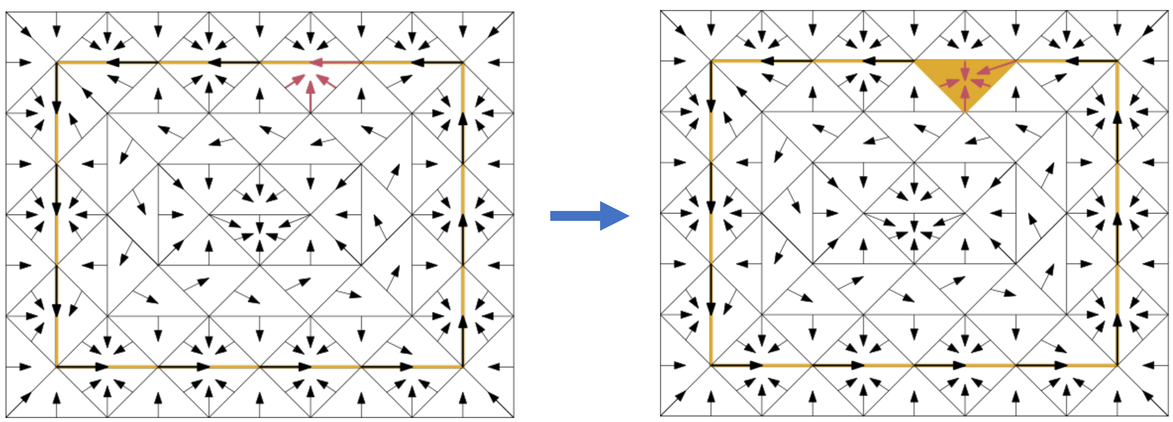
Tracking with Continuation: Case 3

Theorem[DLMS22]: Let S denote an isolated invariant set under $\mathcal{V}$, and let $\mathcal{V}'$ denote an atomic coarsening of $\mathcal{V}$ where the unique bifurcating multivector does not intersect S. Then $\mathsf{Inv}_{\mathcal{V}'}(\mathsf{S})$ is an invariant set and $(\mathsf{cl}(\mathsf{S}), \mathsf{mo}(\mathsf{S}))$ is an index pair for both.

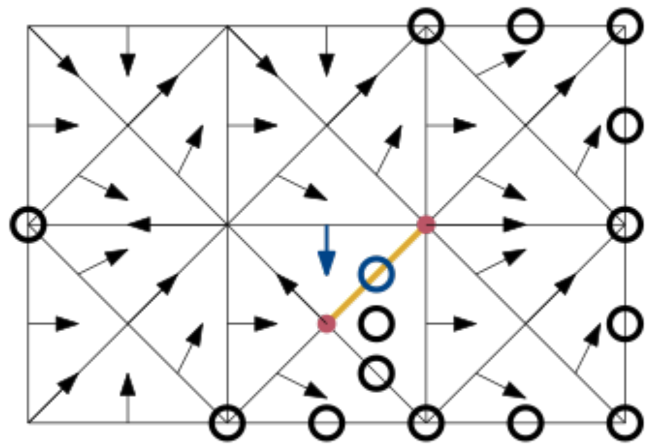


Tracking with Continuation: Case 4

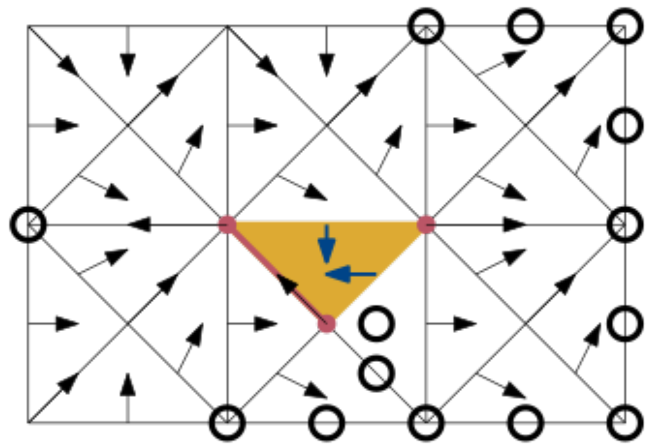
More complicated. Bifurcating vector intersects S but isn't contained in S. Let $A = \langle S \cup V \rangle_{\mathcal{V}}$ denote the minimal $\mathcal{V}$-compatible, convex set containing S.



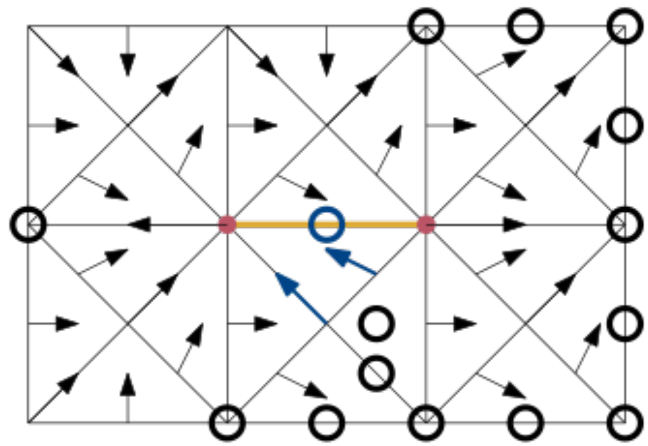
Tracking Example



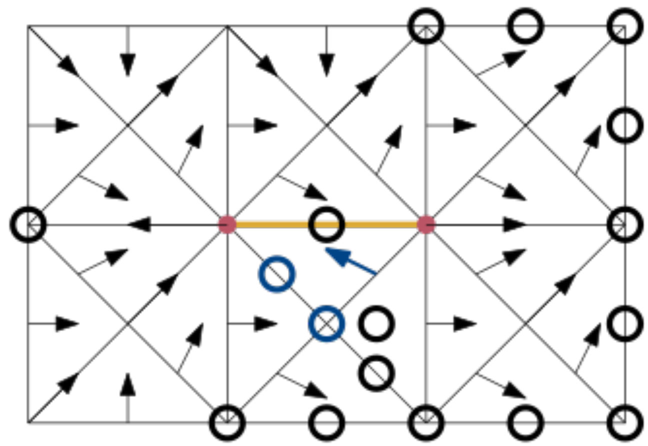
Tracking Example



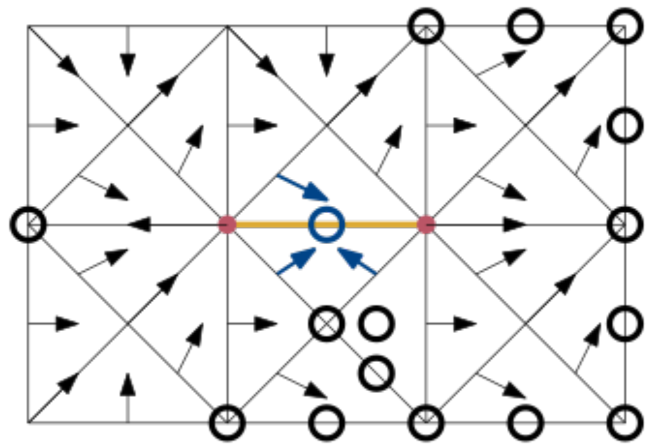
Tracking Example



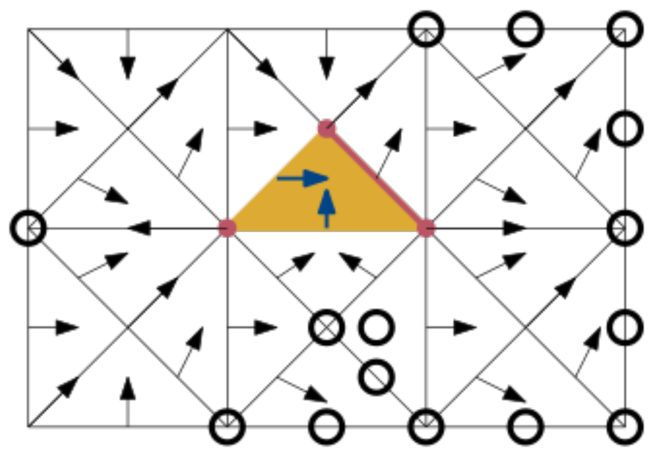
Tracking Example



Tracking Example



Tracking Example



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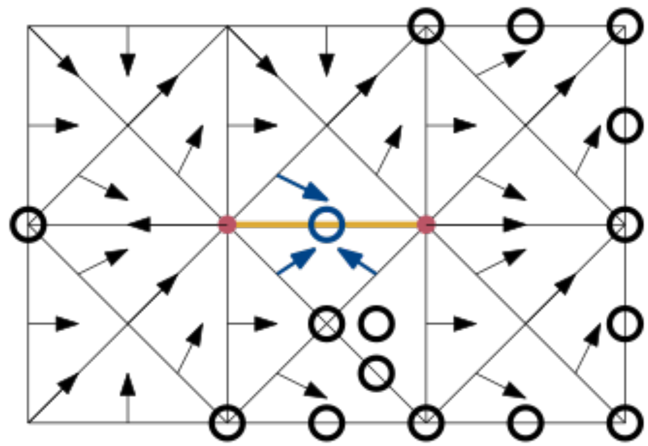

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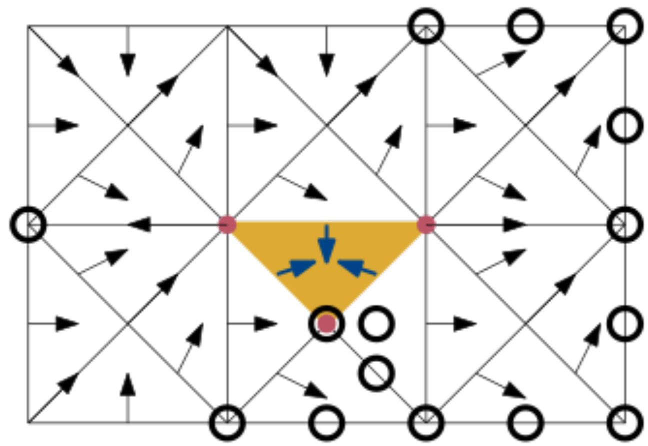
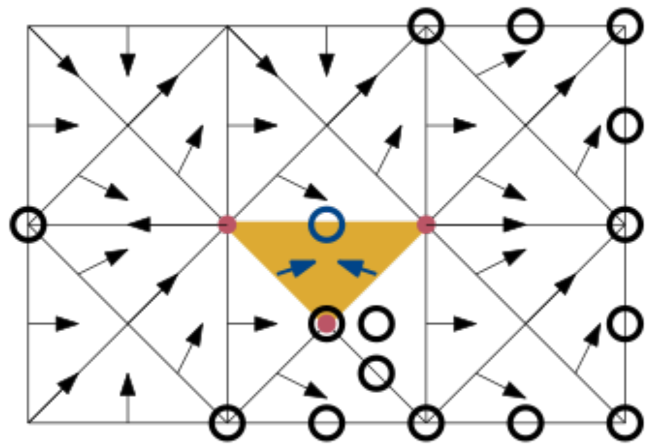


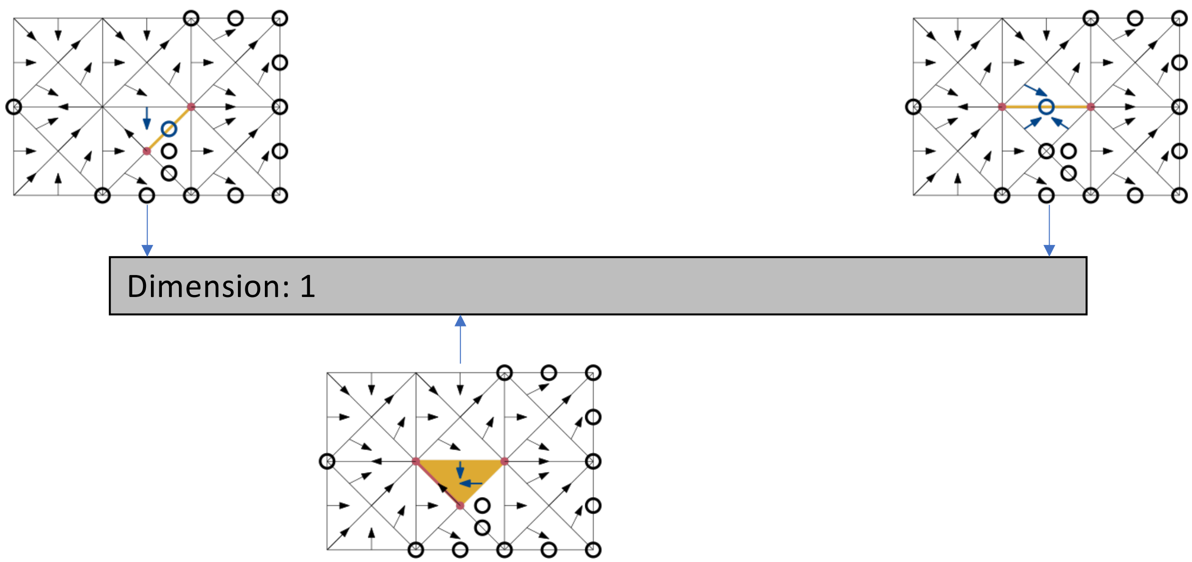
Continuation breaks!!!

Tracking Example



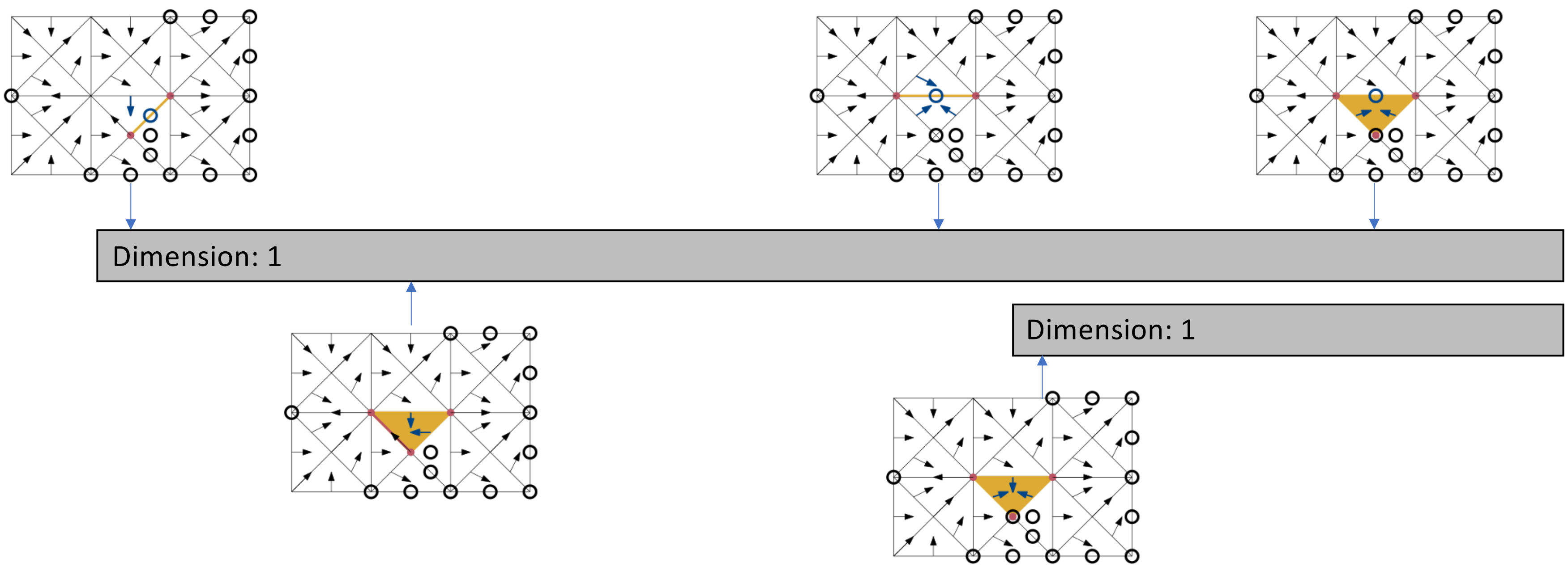
Solution: Persistence

Theorem[DLMS22]: A sequence of index pairs corresponding to a continuation can be converted to a relative zigzag filtration with a “static” barcode.



Solution: Persistence

If S and S' don't share an index pair when using case 4, compute persistence instead!



Conclusion & Future Work

- In this presentation: devised method for tracking invariant set. But…
- Order of atomic rearrangements?
- Inference?