# Persistence of the Conley Index in Combinatorial Dynamical Systems

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# Overview & Outline

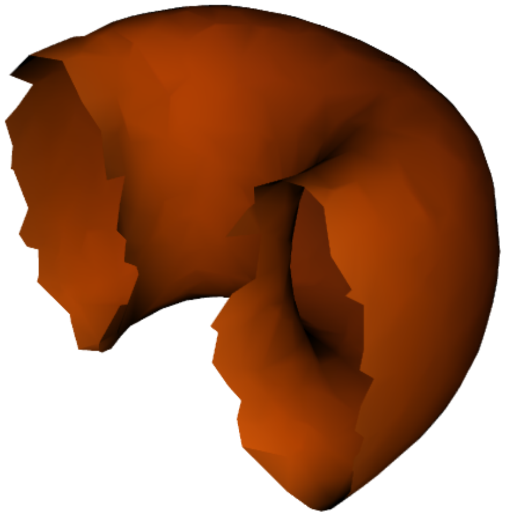
- Persistence
- Combinatorial Dynamical Systems & Conley Index
- Capturing changes in Dynamical Systems via Persistence

# Persistent Homology

Summarizes changing homology of a filtration [ELZ00]

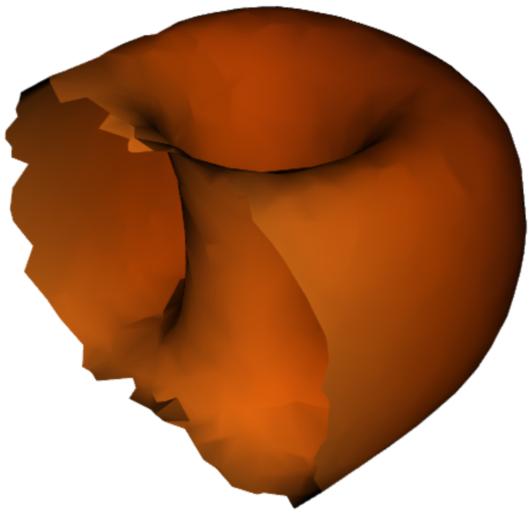
$$K_1 \subseteq K_2 \subseteq \ldots \subseteq K_n = K$$

# Persistence Example



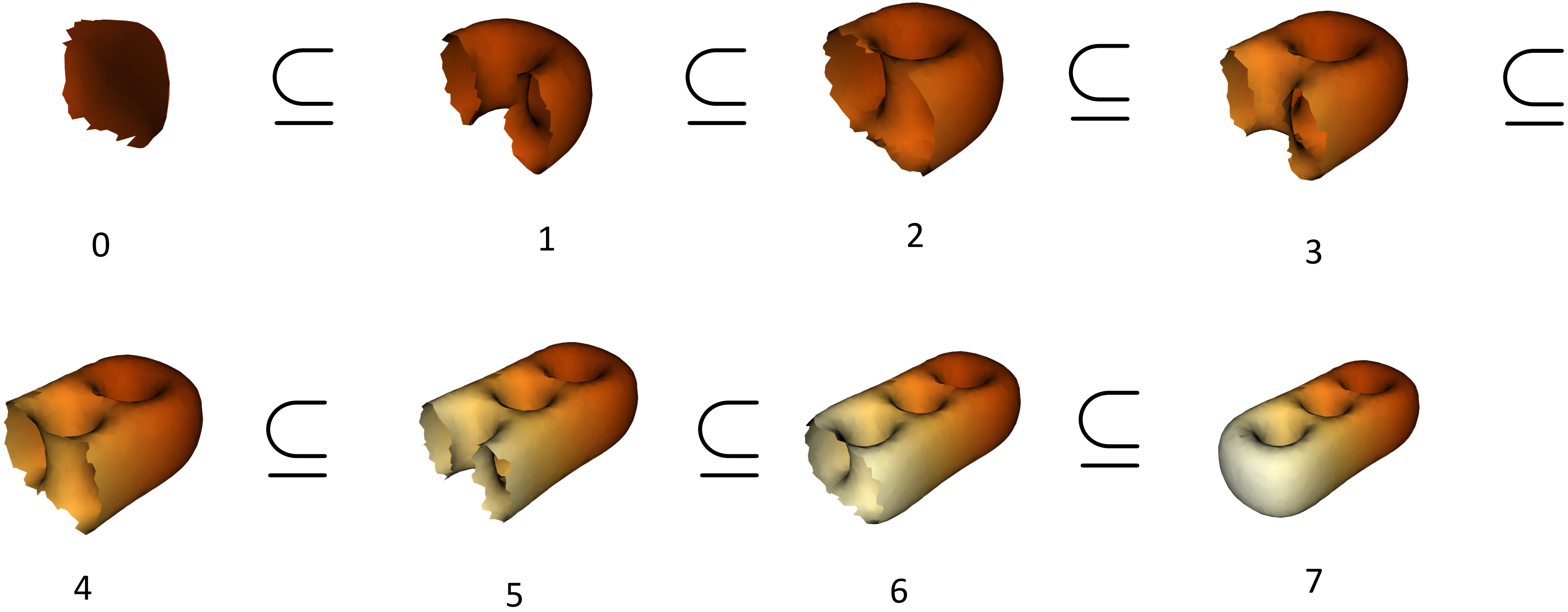
1

$\subseteq$

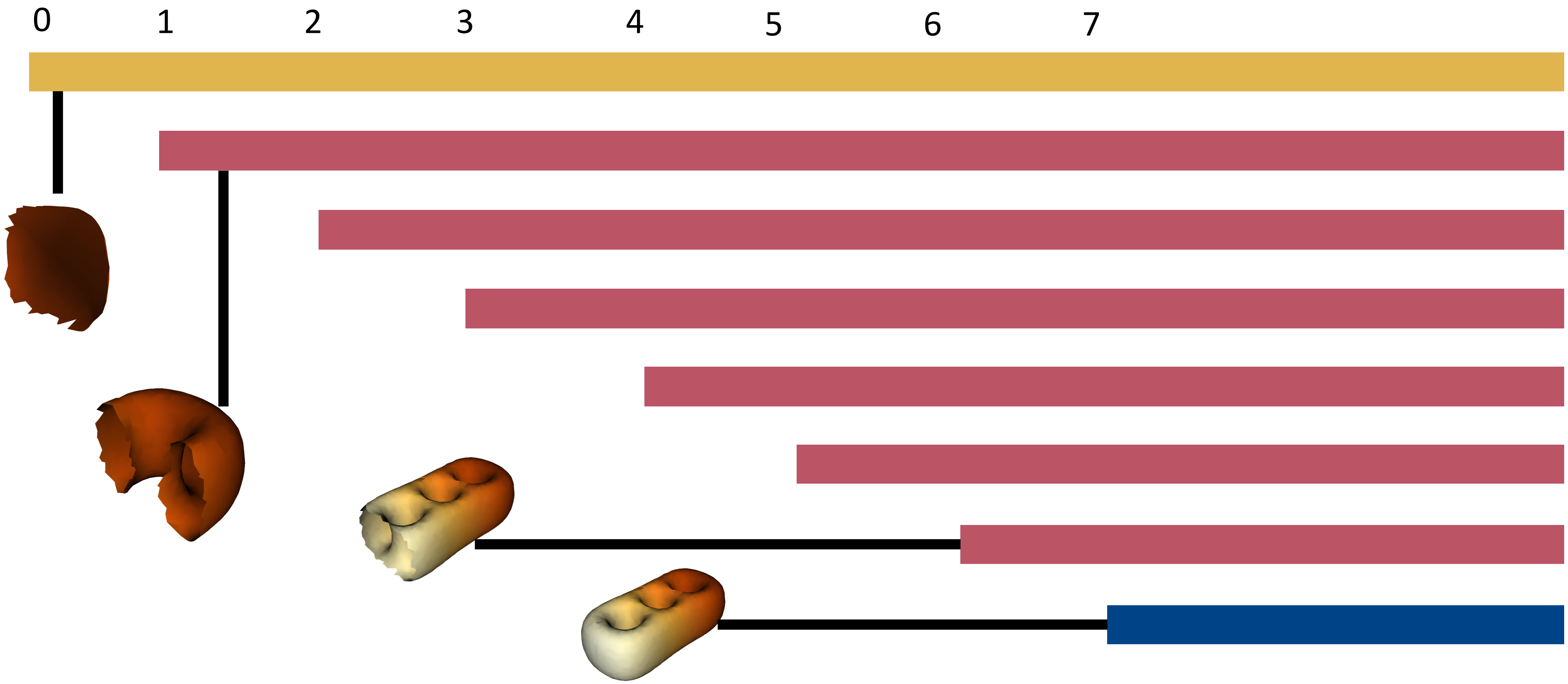


2

# Persistence Example

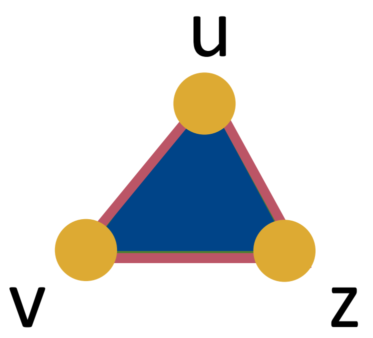


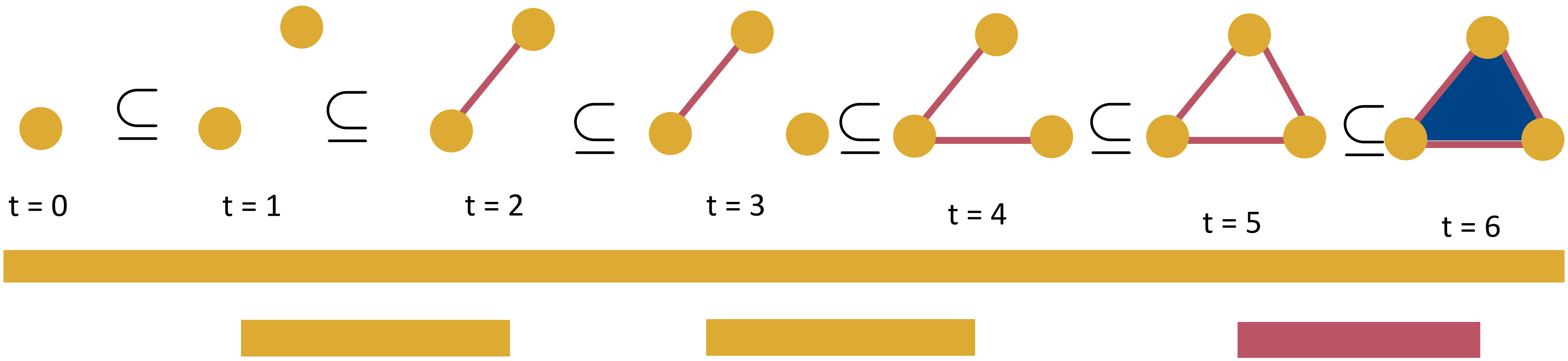
# Persistence Example



# Persistence Example

K =

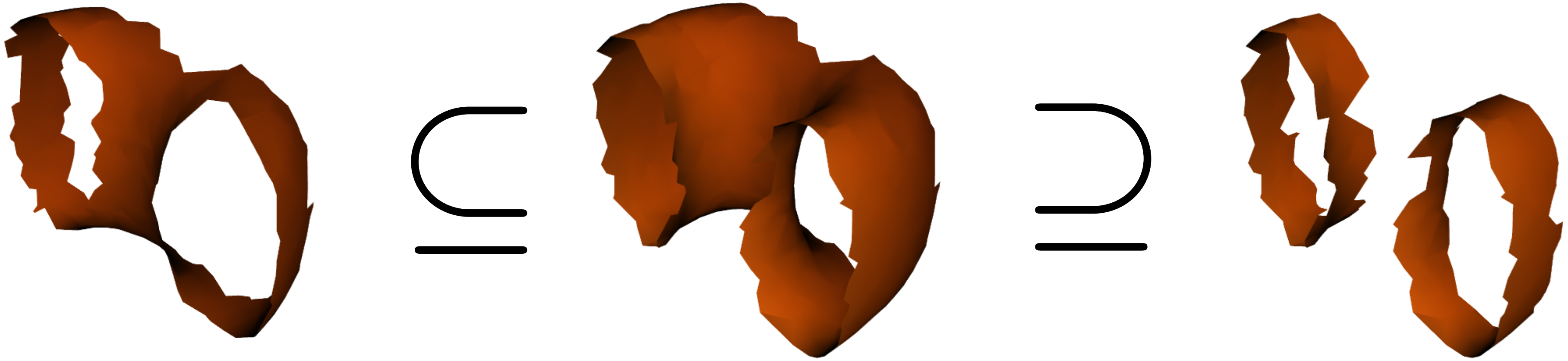


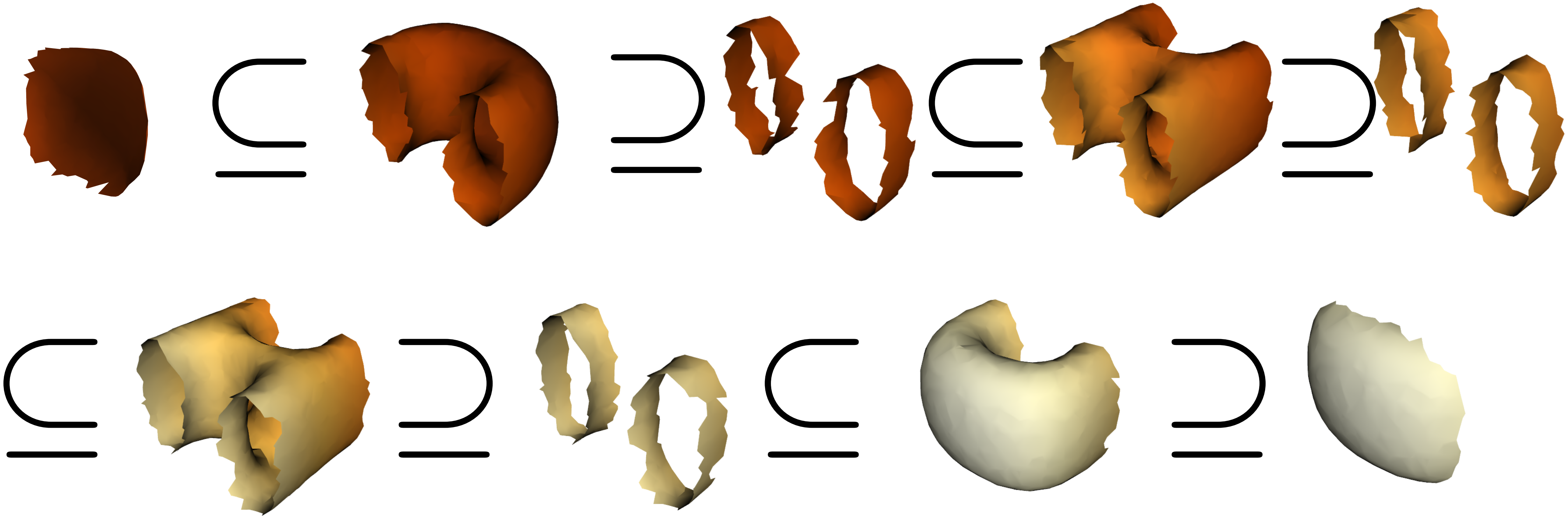

$$K_0 \subset K_1 \subset K_2 \subset K_3 \subset K_4 \subset K_5 \subset K_6$$

# Zigzag Persistence

$$K_1 \subseteq K_2 \supseteq K_3 \subseteq \ldots \supseteq K_n$$

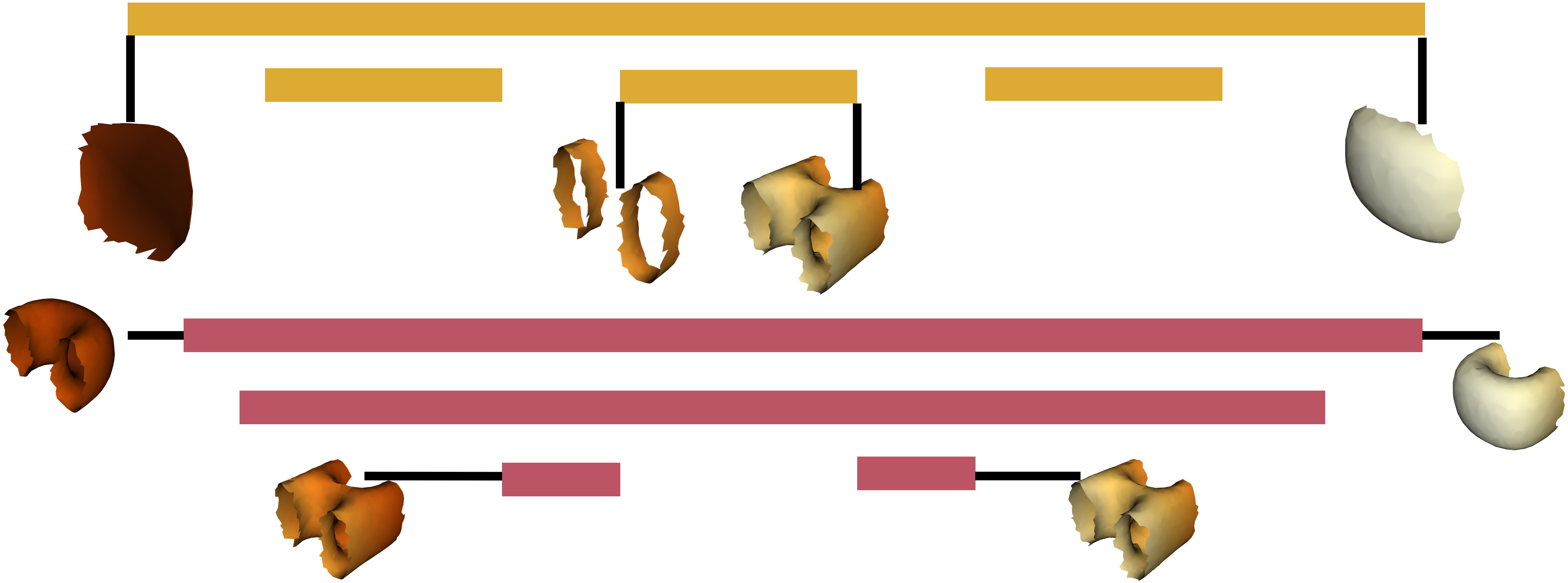


# “Level Set” Persistence



[CDM09] [DW07]

# Level Set Barcode



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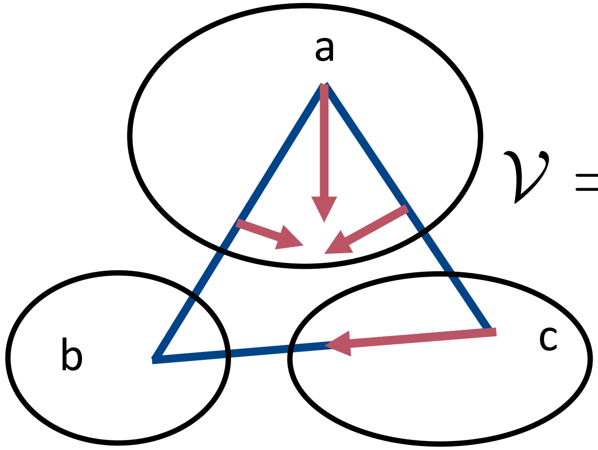
# Multivectors

Let $K$ denote a simplicial complex and $\leq$ denote the face relation.

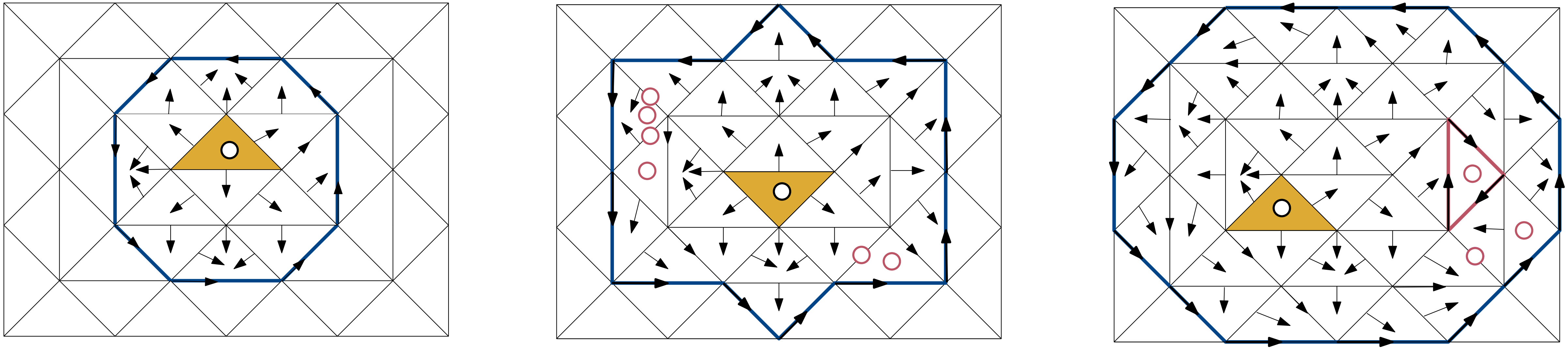
**Definition:** A <u>multivector</u> $V$ is a convex subset of $K$ with respect to $\leq$ .

**Definition:** A <u>multivector field</u> $\mathcal{V}$ is a partition of $K$ into multivectors.




$$\mathcal{V} = \{\{b\}, \{b, bc\}, \{a, ab, ac, abc\}\}$$

# Multivector Fields



# Multivector Fields as a Dynamical System

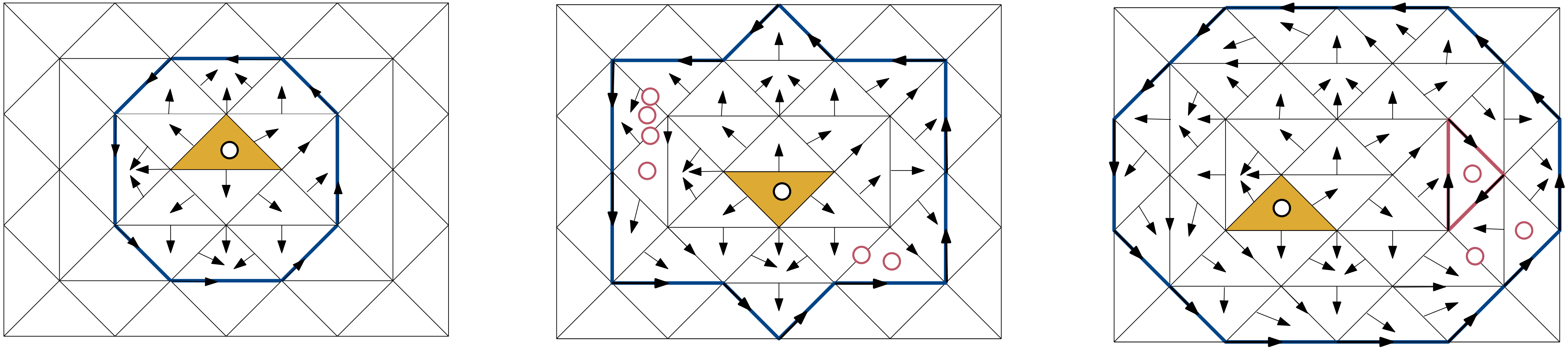
Let $\sigma \in K$. Then $\mathrm{cl}(\sigma) = \{\tau \in K \mid \tau \leq \sigma\}$.

$[\sigma]_{\mathcal{V}}$ denotes the vector in $\mathcal{V}$ containing $\sigma$

Dynamics generator $F_{\mathcal{V}} : K \multimap K$ defined as:

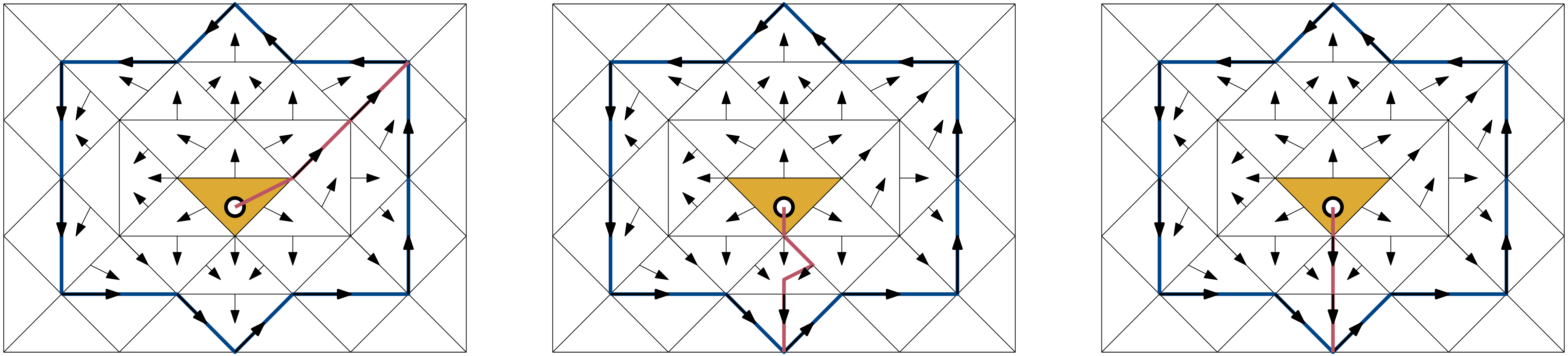
$$F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$$

# Combinatorial Dynamical Systems



# Paths

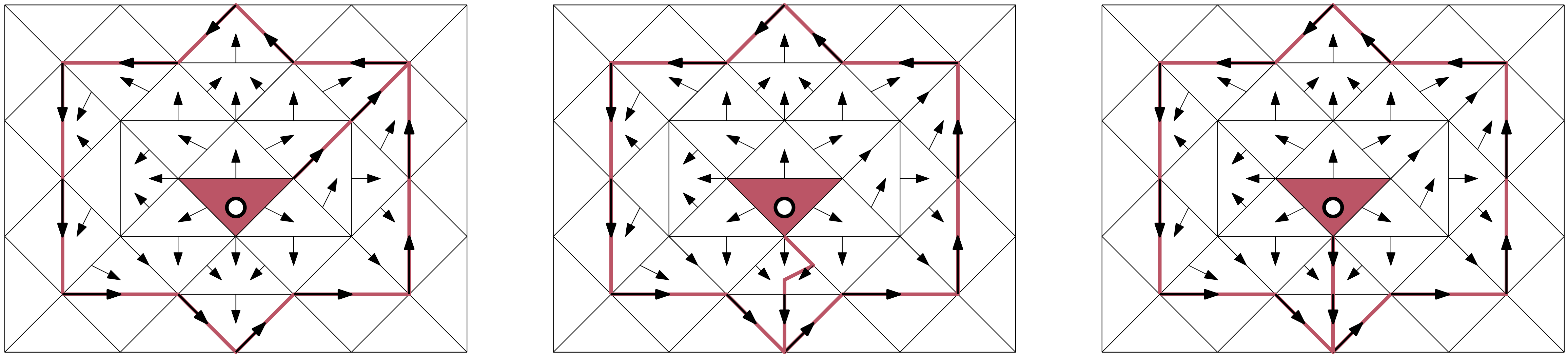
**Definition:** A path is a finite sequence of simplices $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



# Solutions

**Definition:** A solution is a bi-infinite sequence of simplices

$\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ such that $\sigma_{i+1} \in F_{\mathcal{V}}(\sigma_i)$



# Solutions

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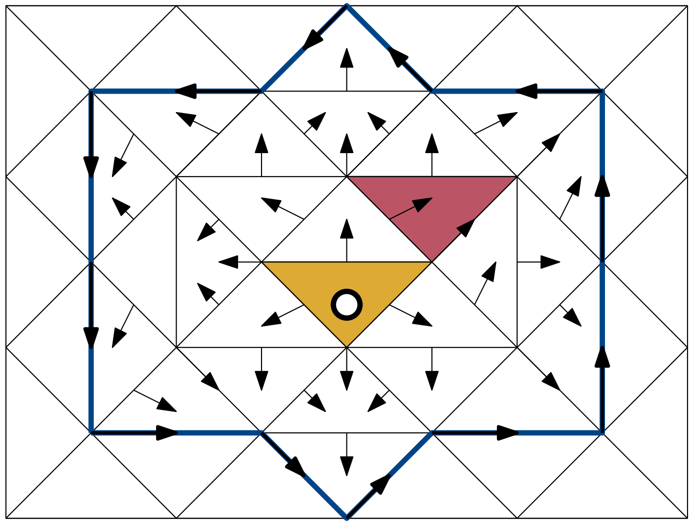


But as $F_{\mathcal{V}}(\sigma) = [\sigma]_{\mathcal{V}} \cup \mathrm{cl}(\sigma)$, every simplex gives a solution!

# Critical Multivectors

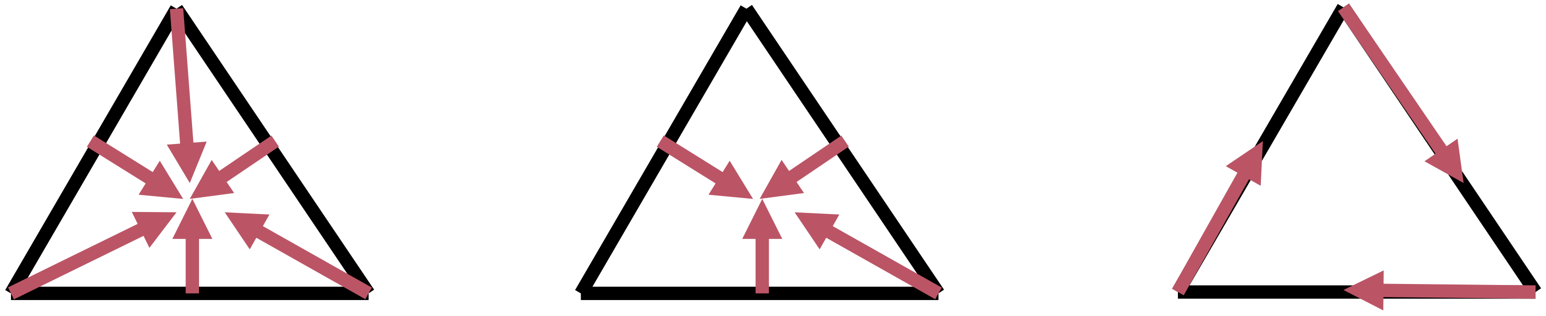
**Definition:** Let $A \subseteq K$. The mouth of A is defined as

$$\mathsf{mo}\,(A) := \mathsf{cl}\,(A) \setminus A$$

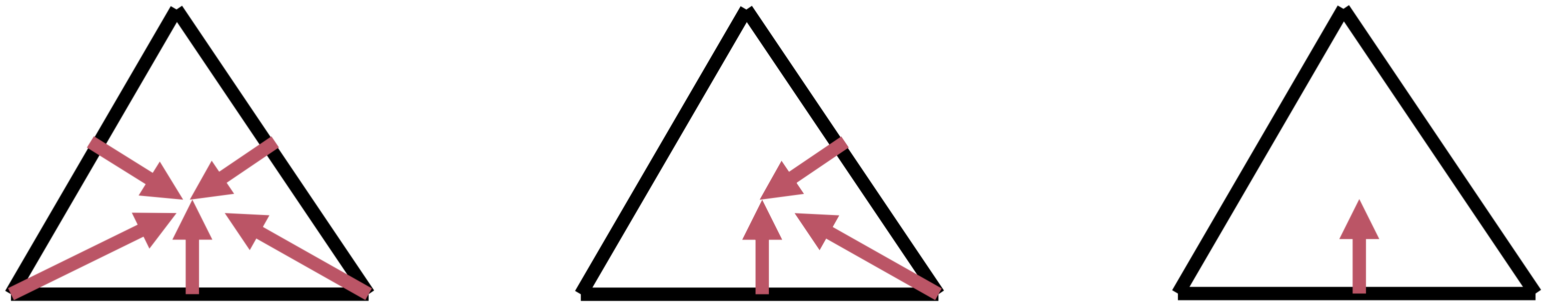
**Definition:** A multivector $[\sigma]_{\mathcal{V}}$ is critical if there exists a k such that $H_k(\mathsf{cl}\,([\sigma]_{\mathcal{V}})\,, \mathsf{mo}\,([\sigma]_{\mathcal{V}}))$ is nontrivial.

# Critical Multivectors

Critical:

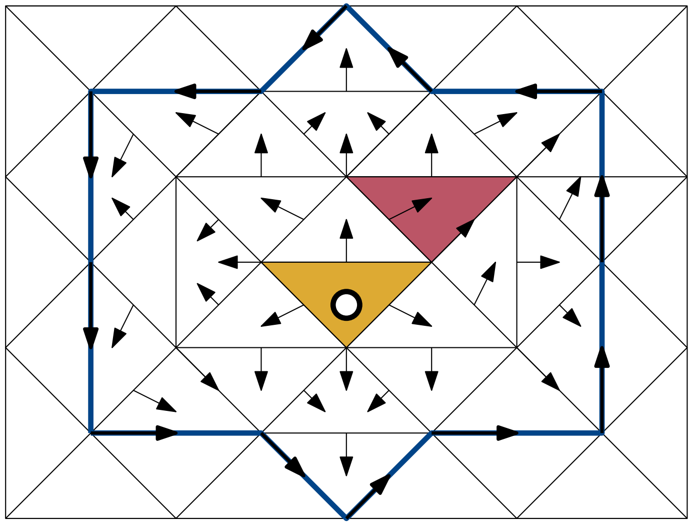


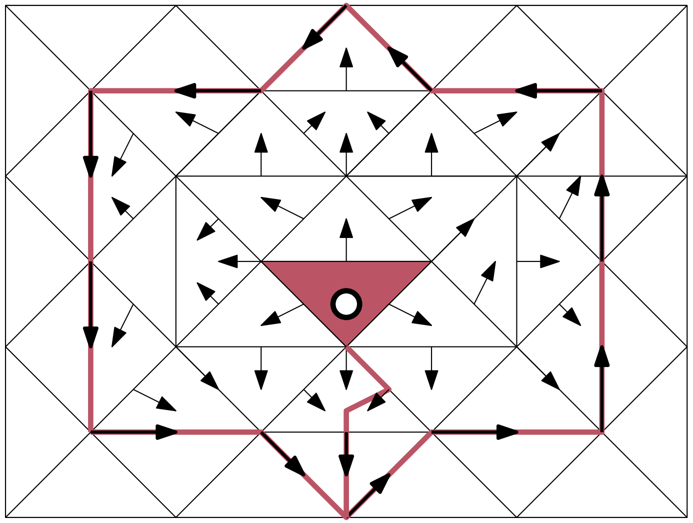
Regular:



# Essential Solutions

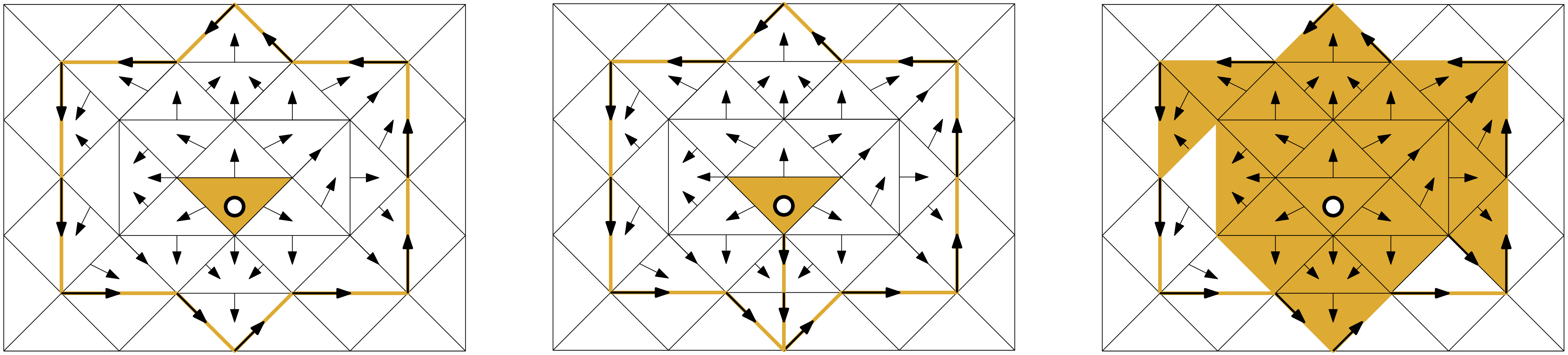
**Definition:** Let $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ denote a solution. If for each $\sigma_i$ where $[\sigma_i]_{\mathcal{V}}$ is noncritical, there exists a $j > i$ and $j' < i$ where $[\sigma_i]_{\mathcal{V}} \neq [\sigma_j]_{\mathcal{V}}$ and $[\sigma_i]_{\mathcal{V}} \neq [\sigma_{j'}]_{\mathcal{V}}$,
then $\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots$ is an essential solution.





# Invariant Sets

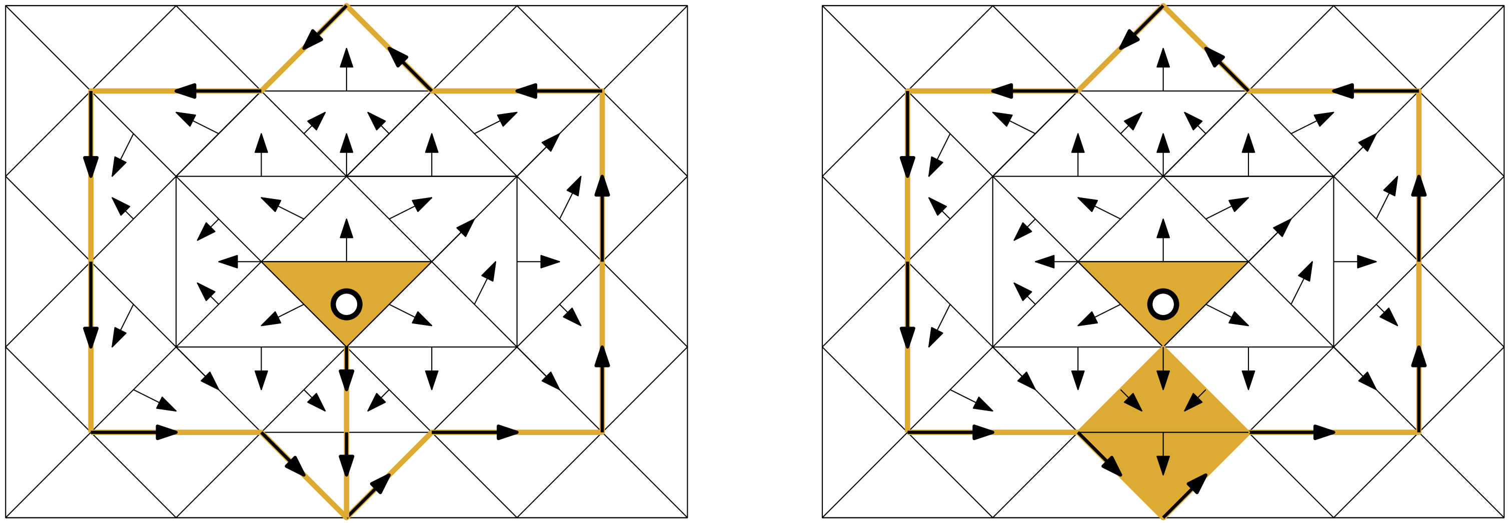
**Definition:** Let $A \subseteq K$. The invariant part of $A$, denoted $\mathrm{Inv}(A)$, is the set of simplices in $A$ which appear in an essential solution in $A$.



If $A = \mathrm{Inv}(A)$, then $A$ is an invariant set.

# Invariant Sets

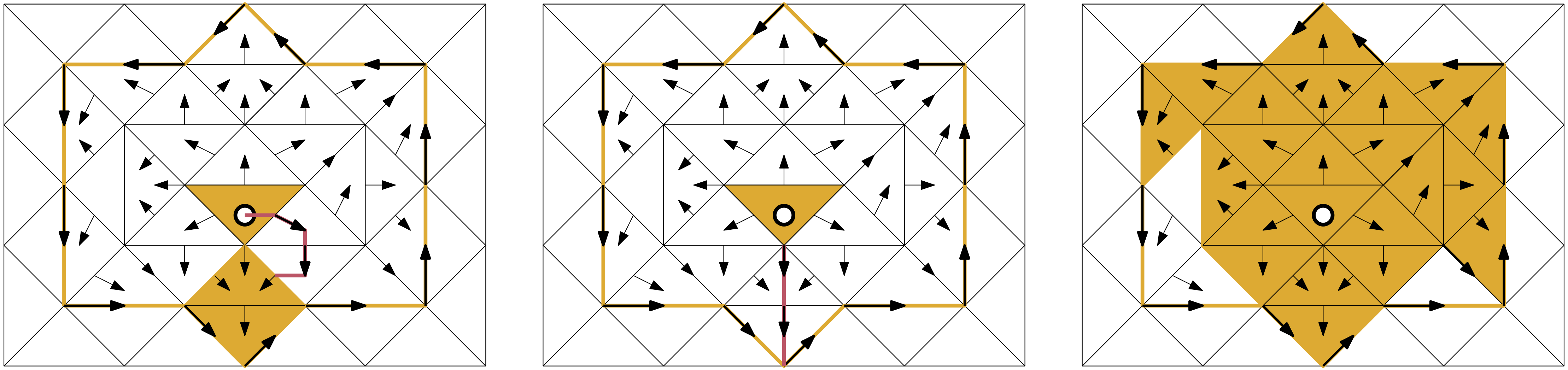
**Definition:** Let $A$ denote an invariant set. If $A$ is equal to a union of multivectors, then $A$ is $\mathcal{V}$-compatible.



From now on, assume that invariant sets are $\mathcal{V}$-compatible.

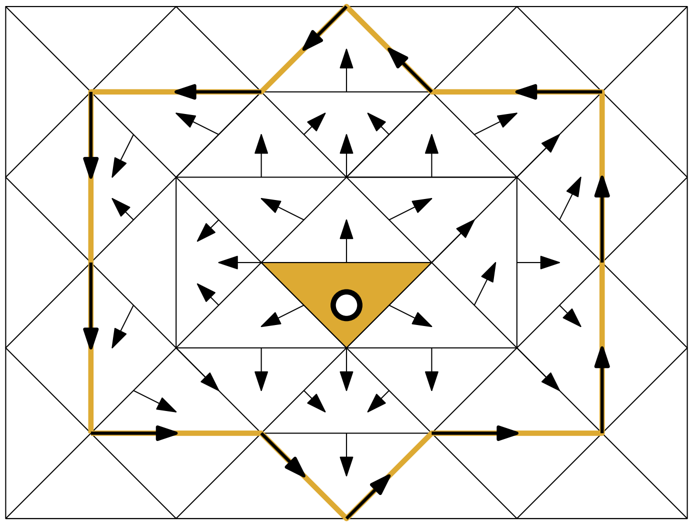
# Isolated Invariant Sets

**Definition:** Let $A \subseteq N \subseteq K$, where $A$ is an invariant set and $N$ is closed (i.e. $N = \mathrm{cl}(N)$). If every path in N with endpoints in $A$ is contained in $A$, then $A$ is an isolated invariant set, and $N$ is an isolating neighborhood for $A$.



# Isolated Invariant Sets

**Note:** We have already seen that the yellow invariant set is not isolated by the rectangle. But does a different neighborhood isolate it?

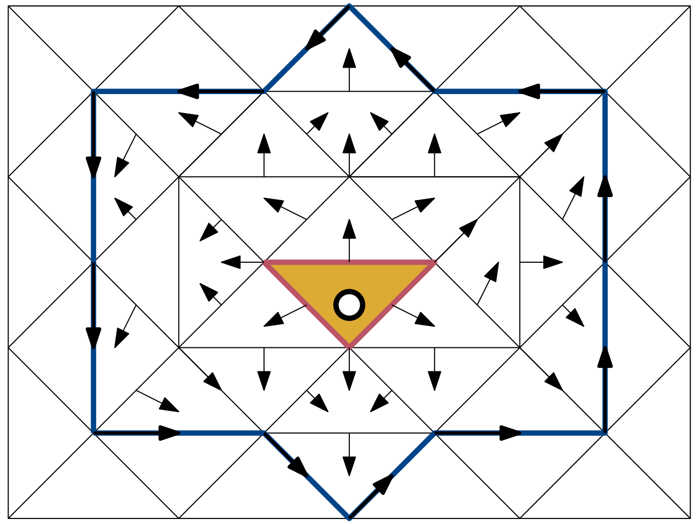


# Index Pairs

**Definition:** Let $A$ be an isolated invariant set, and $E$ and $P$ closed sets such that $E \subseteq P$. If:

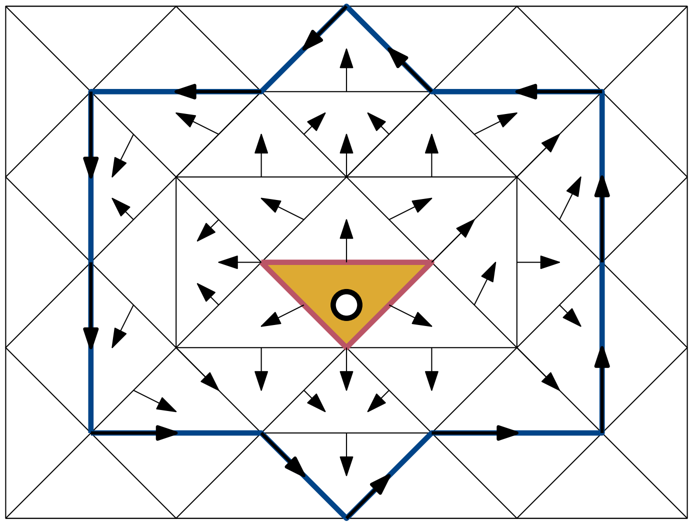
1. $F_{\mathcal{V}}(E) \cap P \subset E$,
2. $F_{\mathcal{V}}(P \setminus E) \subseteq P$, and
3. $A = \mathrm{Inv}(P \setminus E)$

Then $(P, E)$ is an index pair for $A$.

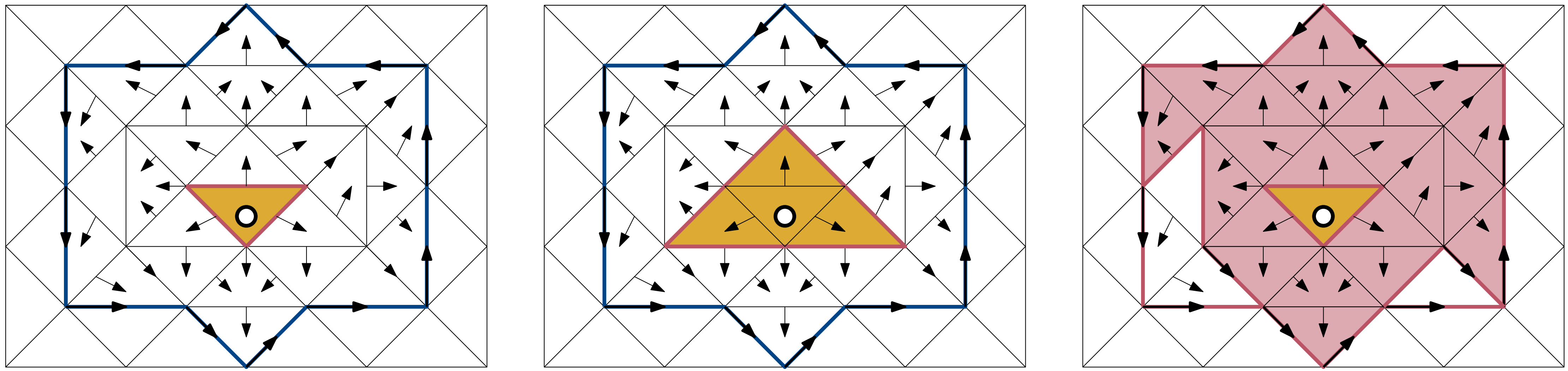


# Conley Index

**<u>Theorem [LKMW2019]:</u>** Let $A$ denote an isolated invariant set. The pair $(\text{cl}(A), \text{mo}(A))$ is an index pair for $A$.



# Index Pairs are Not Unique

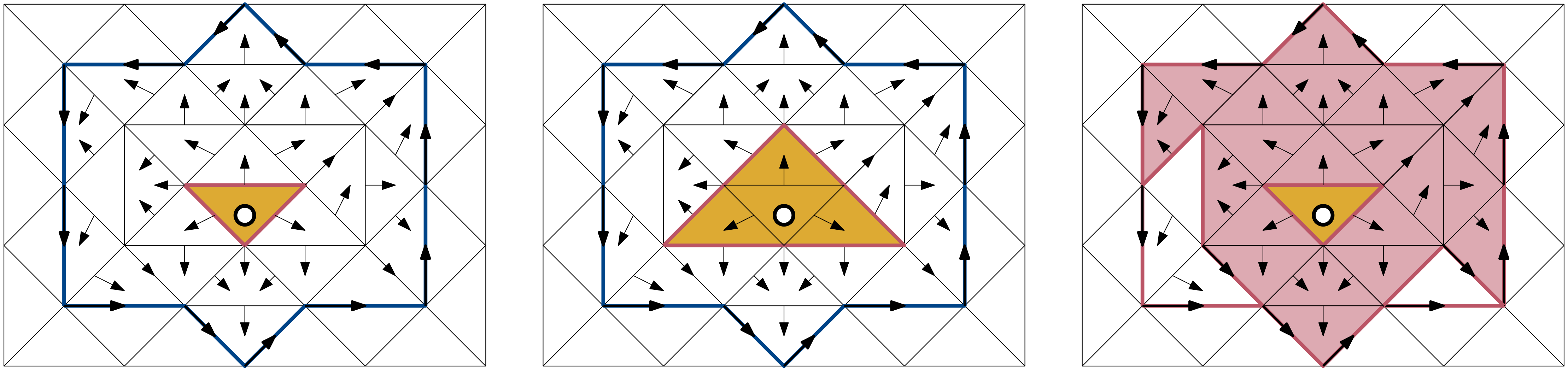


# Conley Index

**Definition:** Let $(P, E)$ be an index pair for $A$. Then the k-dimensional Conley Index is given by $H_k(P, E)$.

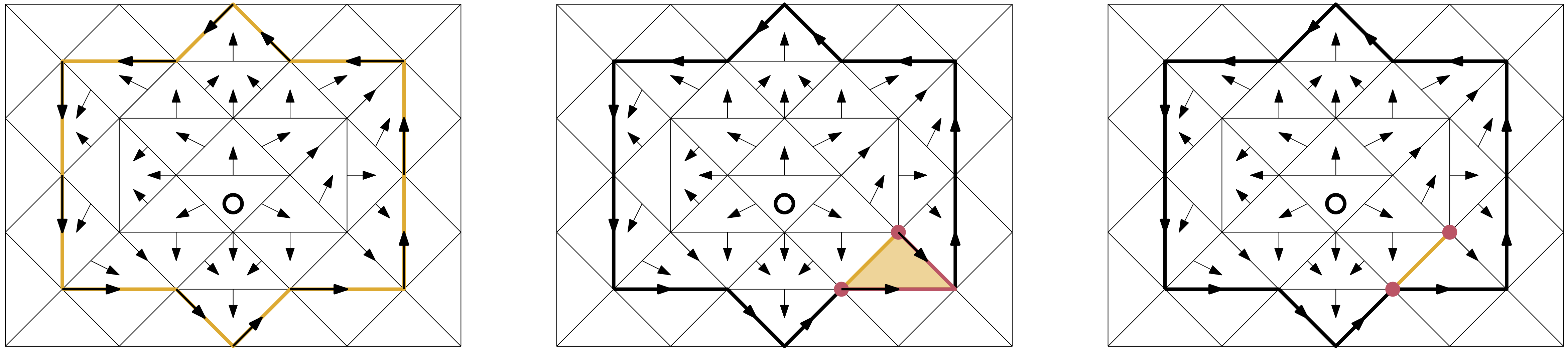
**Theorem [LKMW 2019]:** The k-dimensional Conley Index for $A$ is well defined.

# Conley Indices



$$H_2(R \cup Y, R) = \mathbb{Z}_2$$

# Conley Indices?



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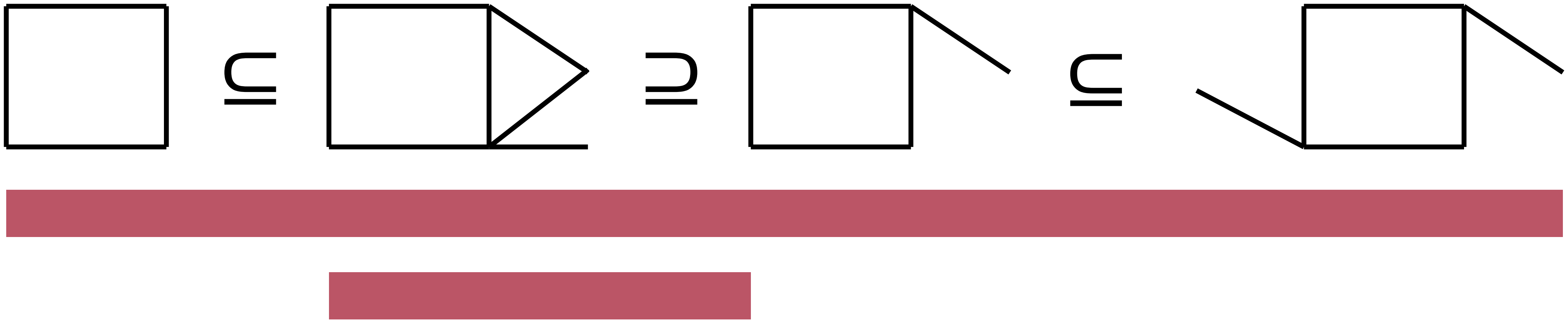
# Overview

Persistence: capture changing homology of spaces



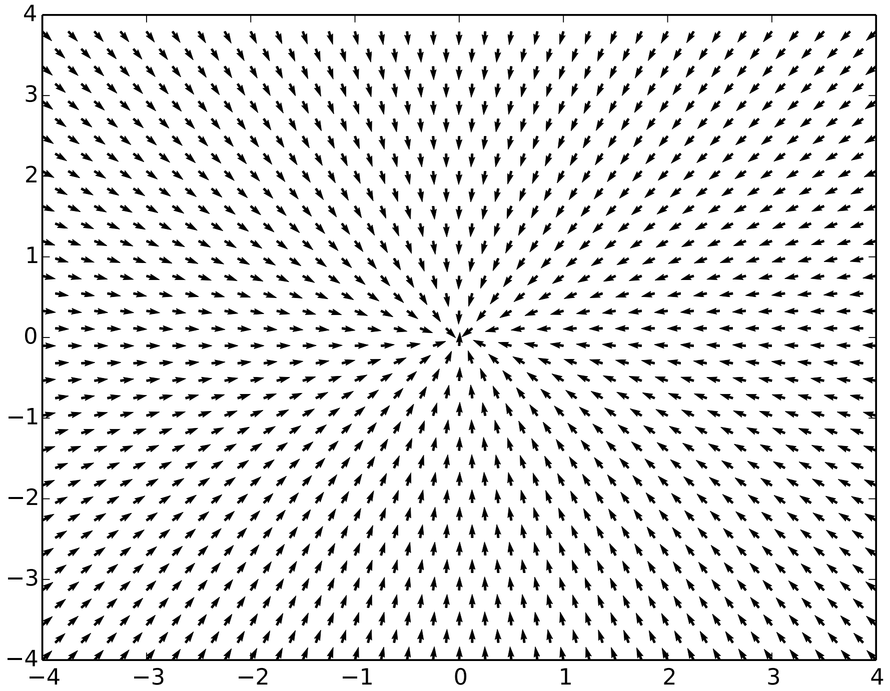
But what about dynamical systems?

# Motivating Example: Hopf Bifurcation

$$x' = -y + x(\lambda - x^2 - y^2)$$

$$y' = x + y(\lambda - x^2 - y^2)$$

# Motivating Example: Hopf Bifurcation




$$\lambda \ll 0$$

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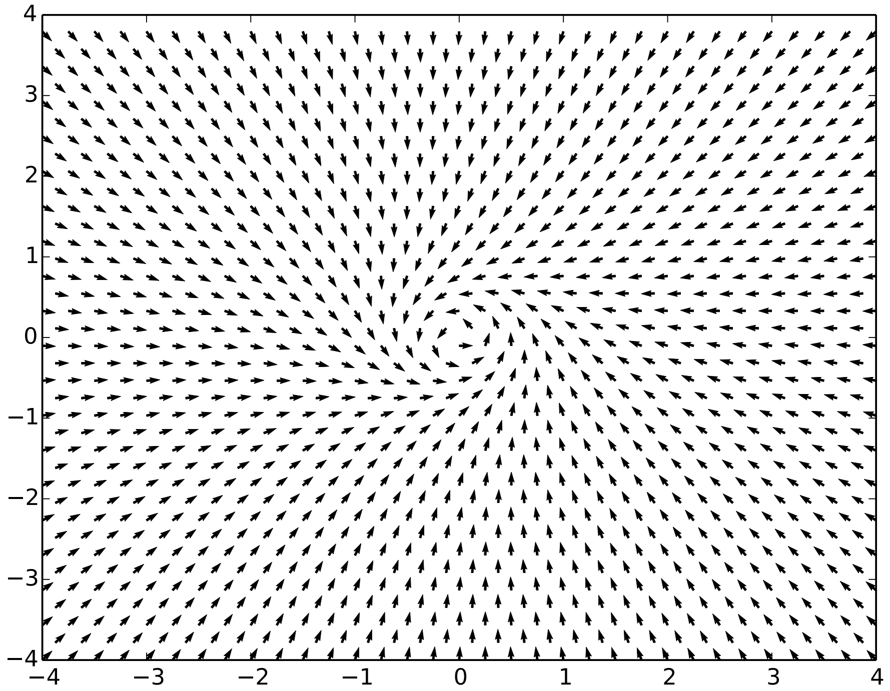


$$\lambda = 0$$

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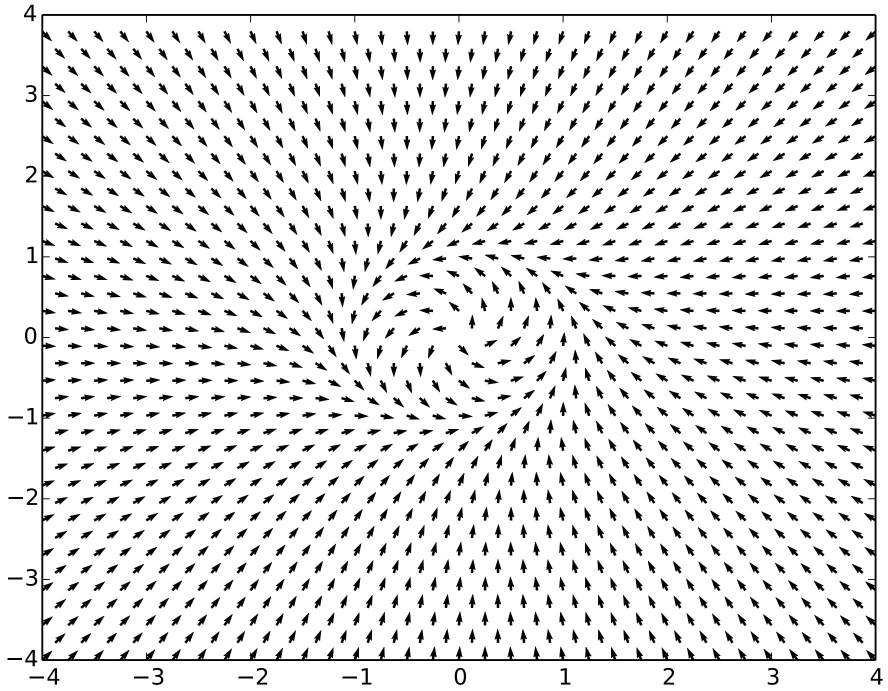
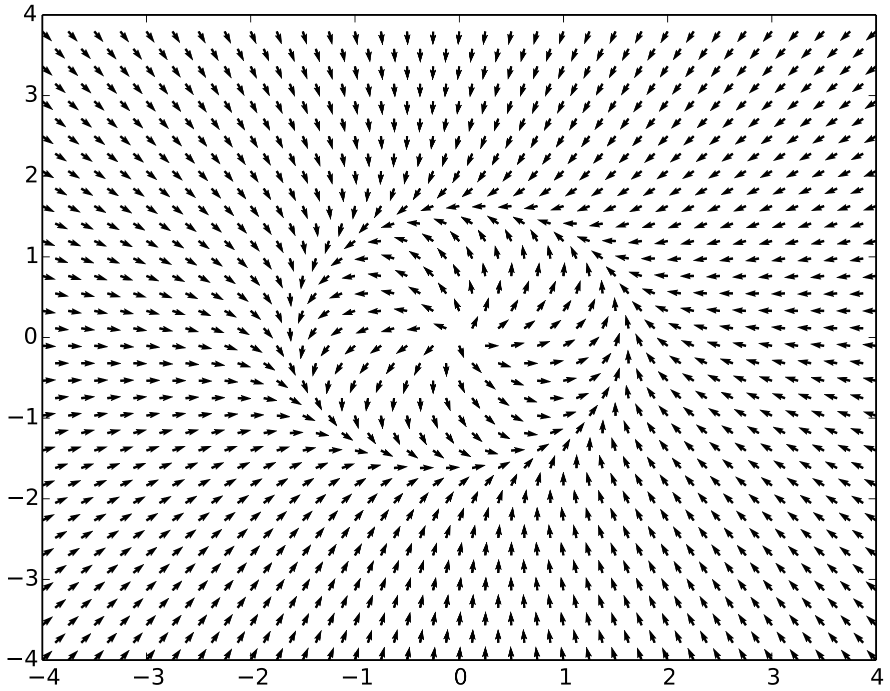


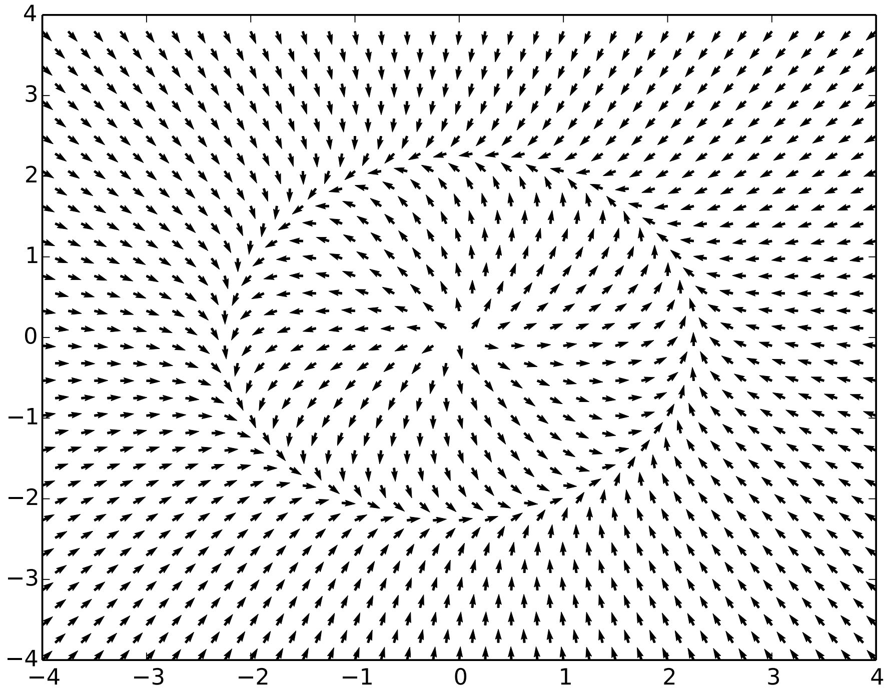

$$\lambda = 1$$

# Motivating Example: Hopf Bifurcation



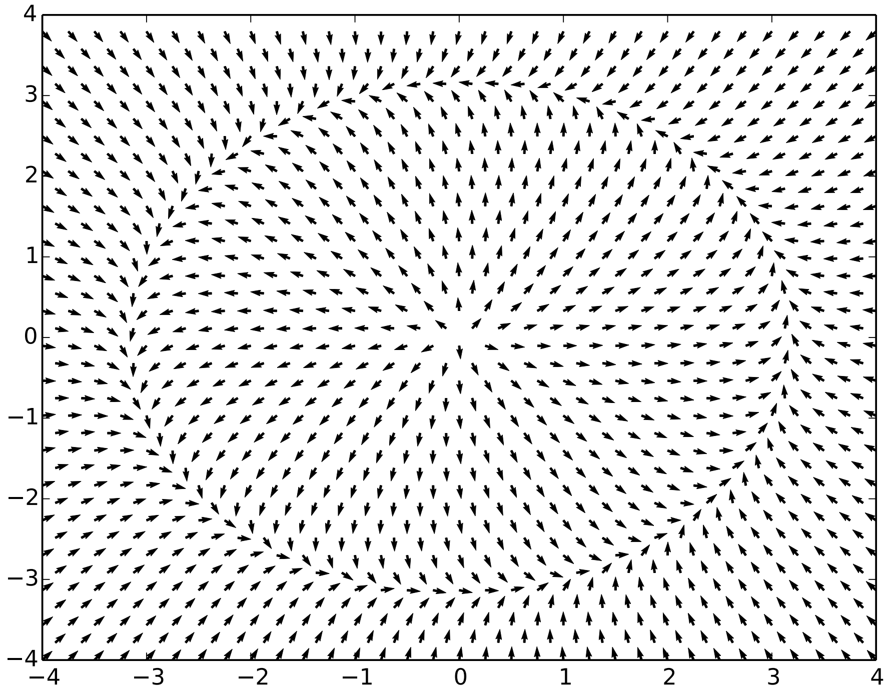

$$\lambda = 2.5$$

# Motivating Example: Hopf Bifurcation



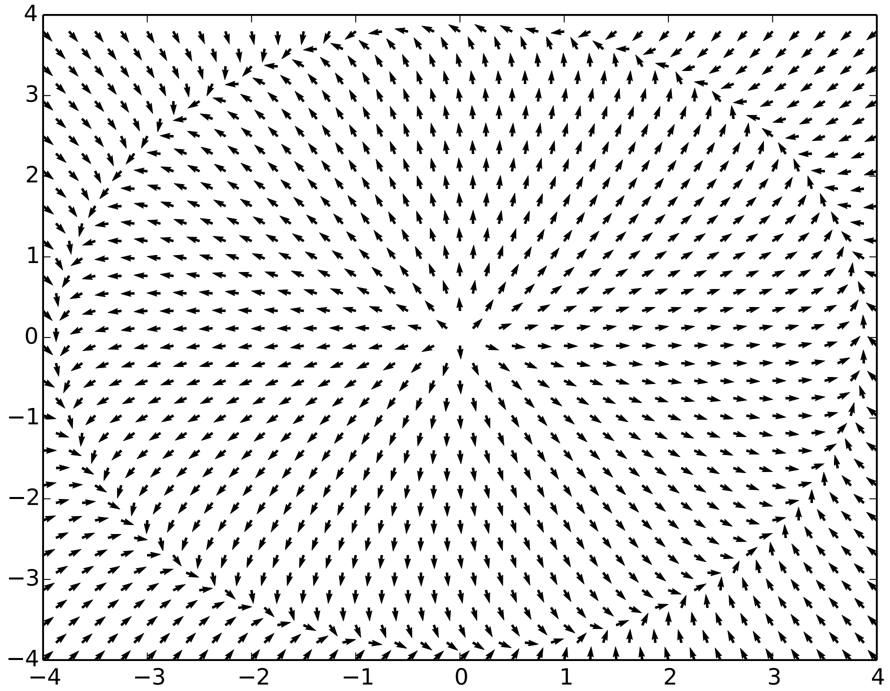

$$\lambda = 5$$

# Motivating Example: Hopf Bifurcation



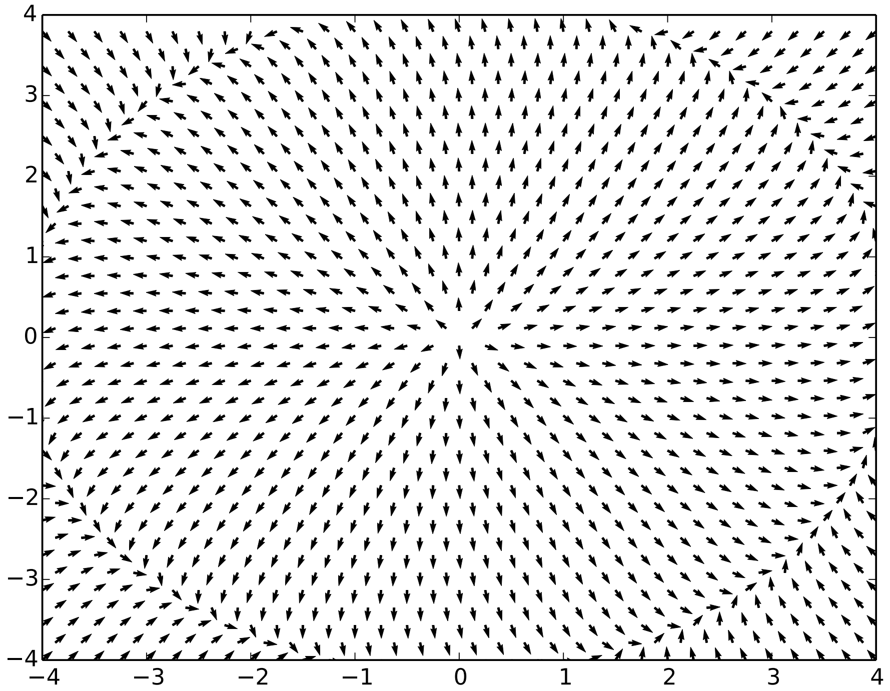

$$\lambda = 10$$

# Motivating Example: Hopf Bifurcation




$$\lambda = 15$$

# Motivating Example: Hopf Bifurcation




$$\lambda = 17.5$$

# Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Can we use persistence to capture this, or a related feature?

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**Yes**, using a special type of index pair

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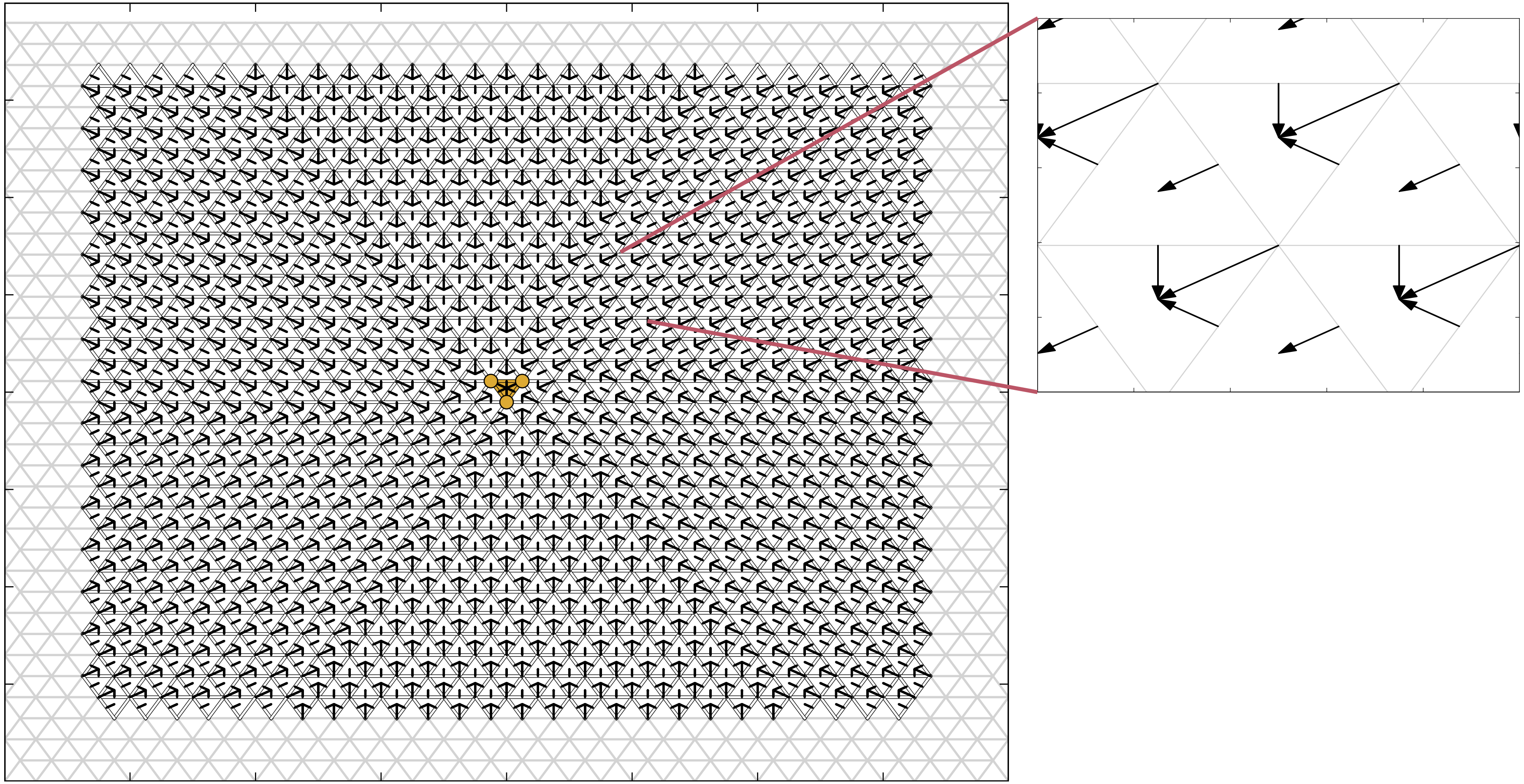

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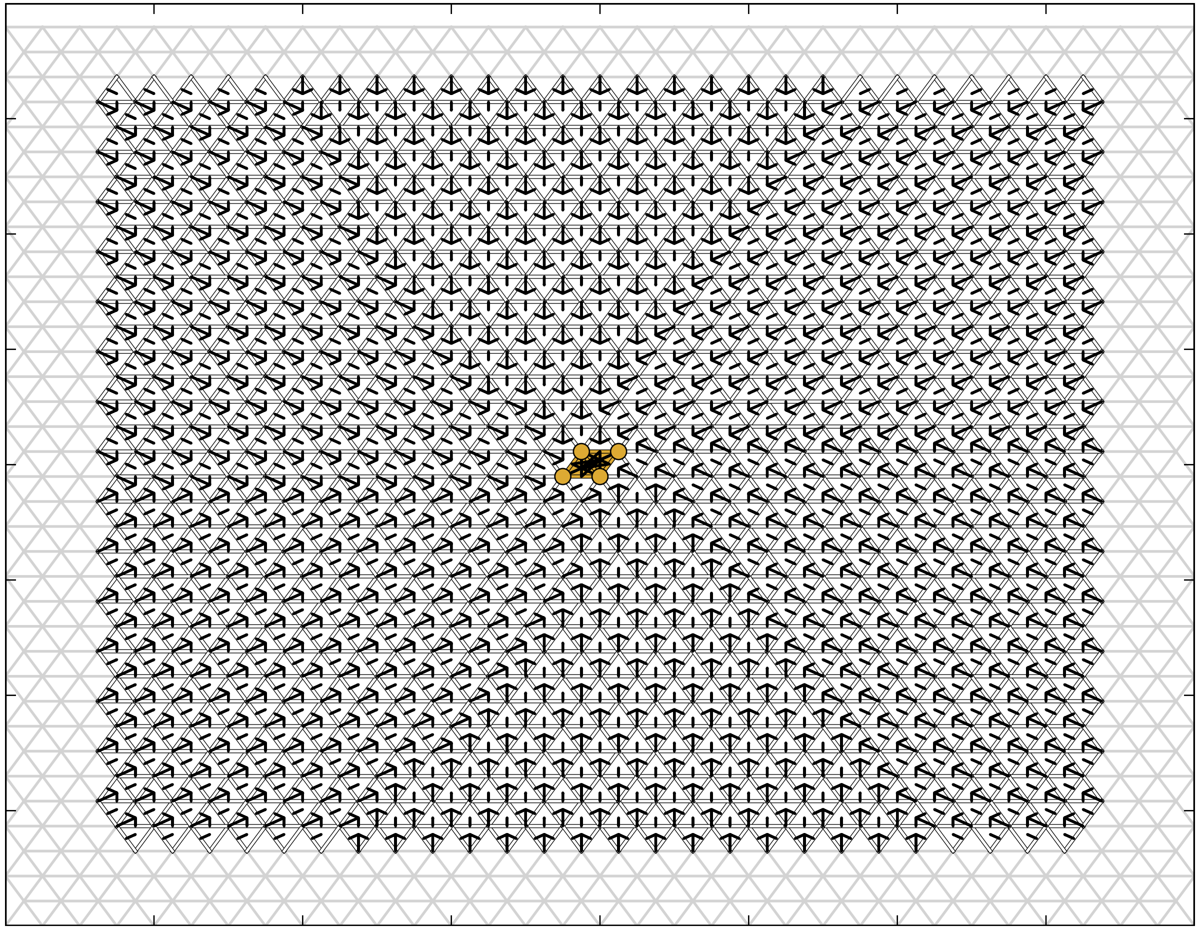

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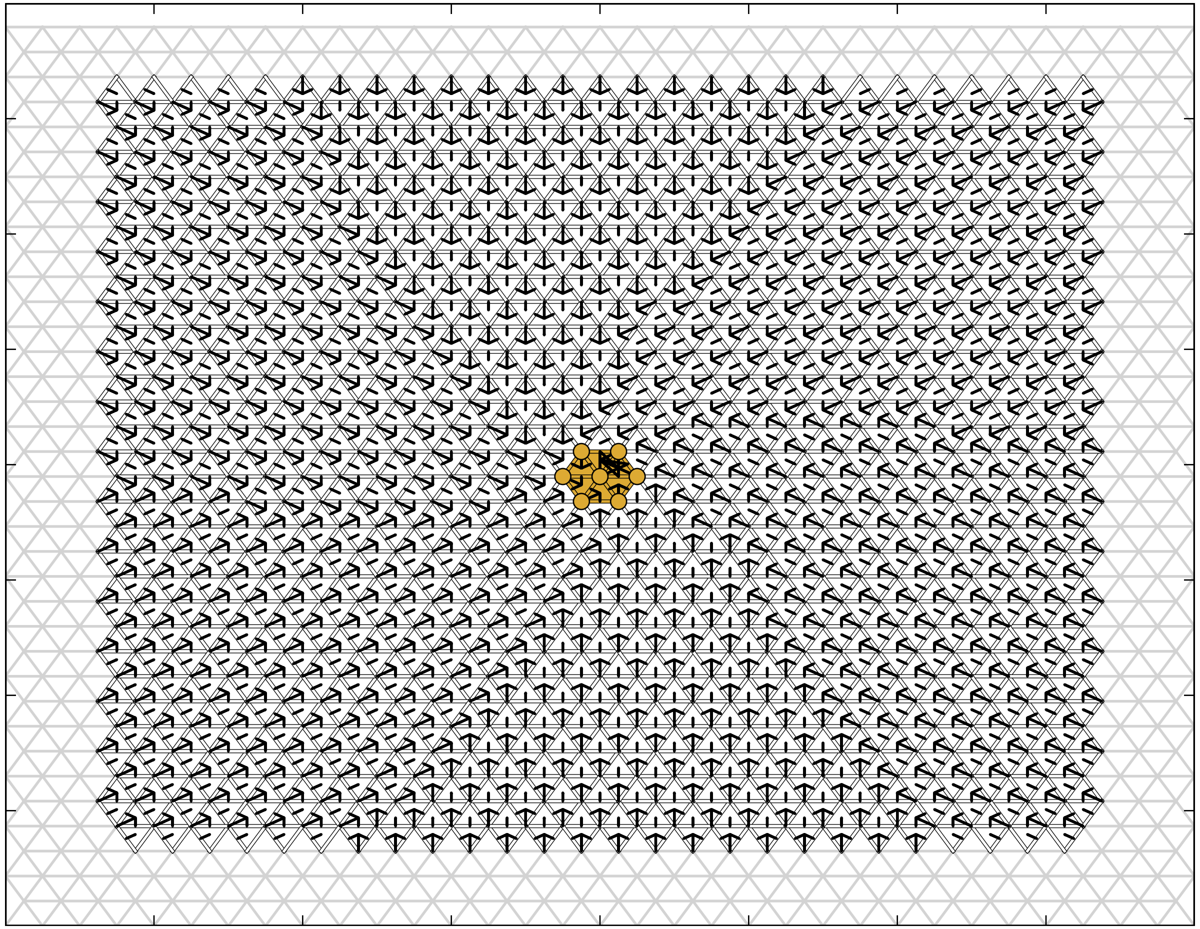

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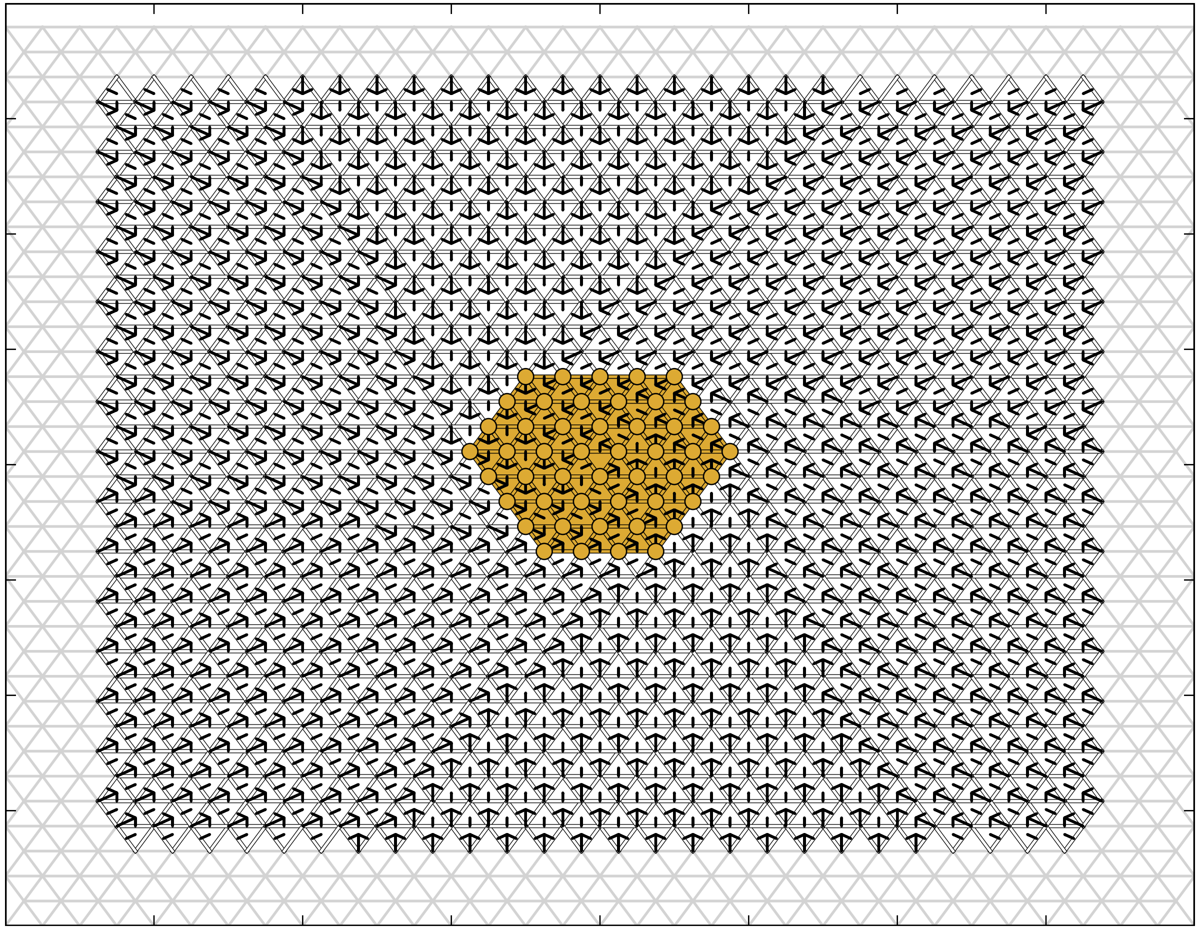

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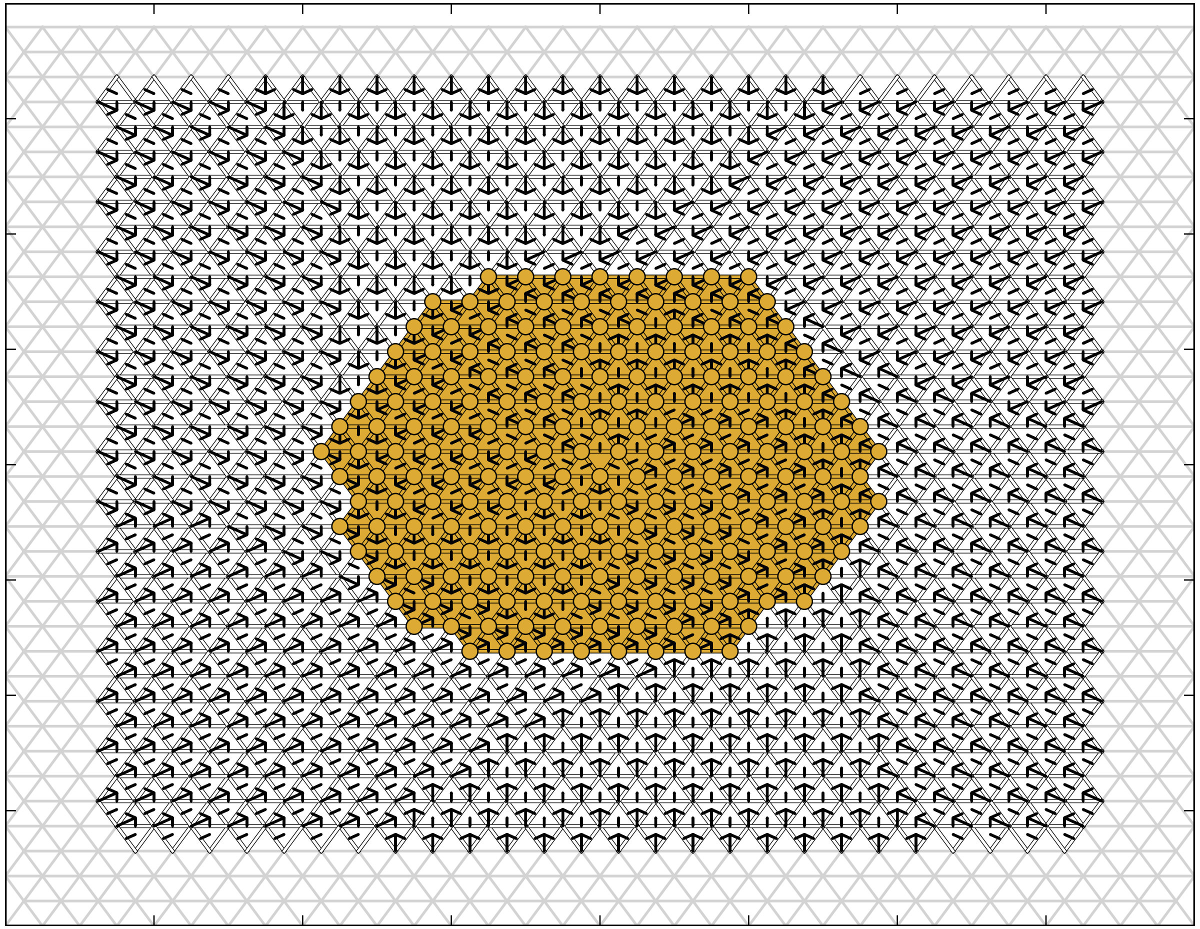

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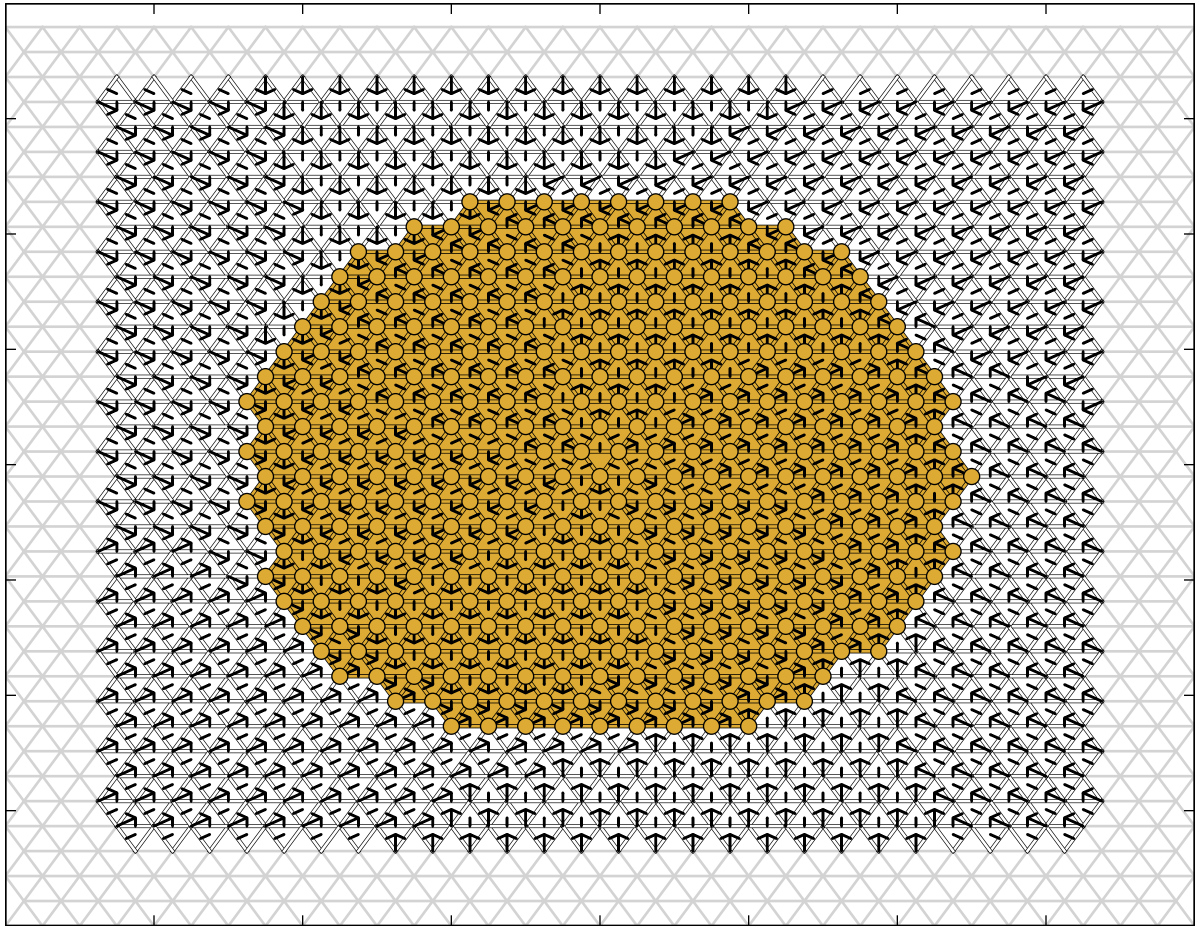

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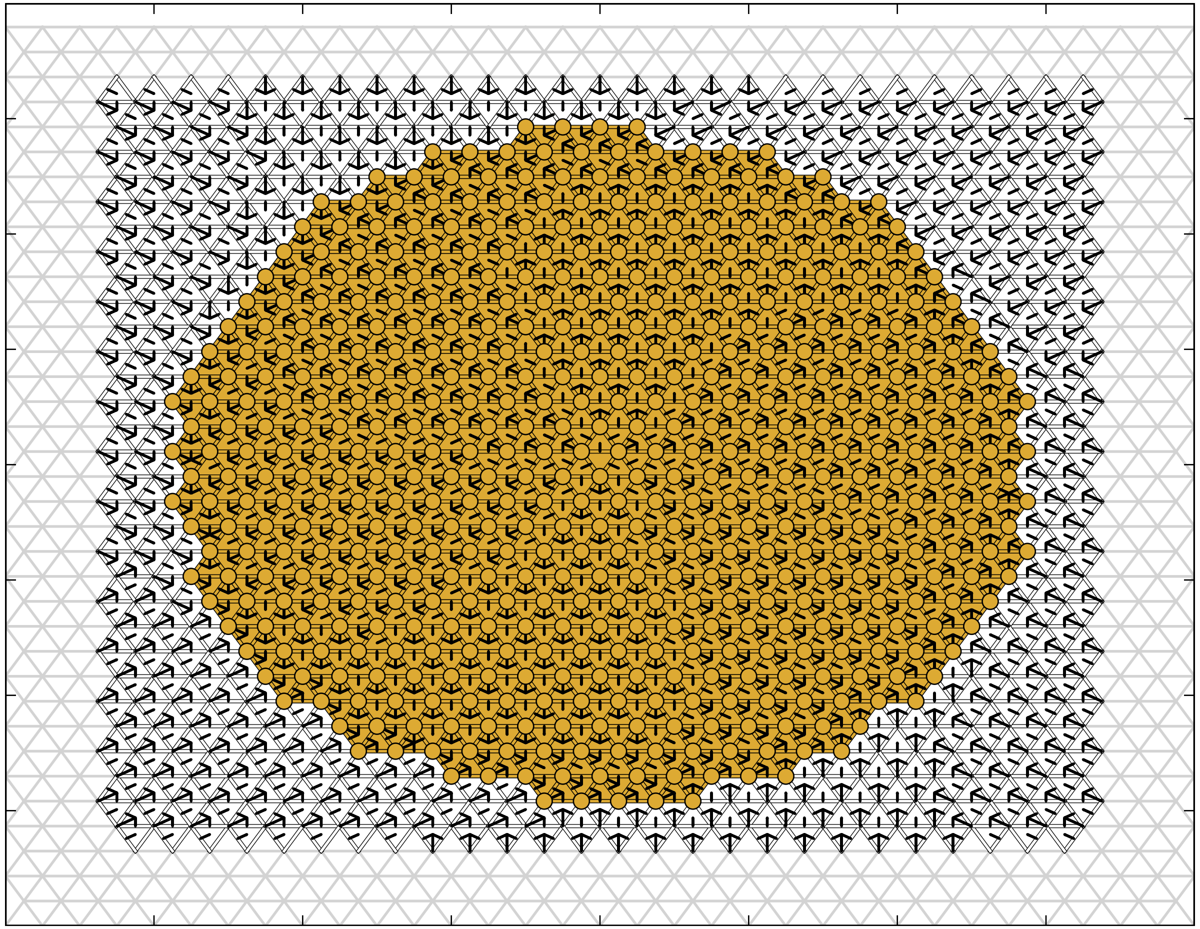

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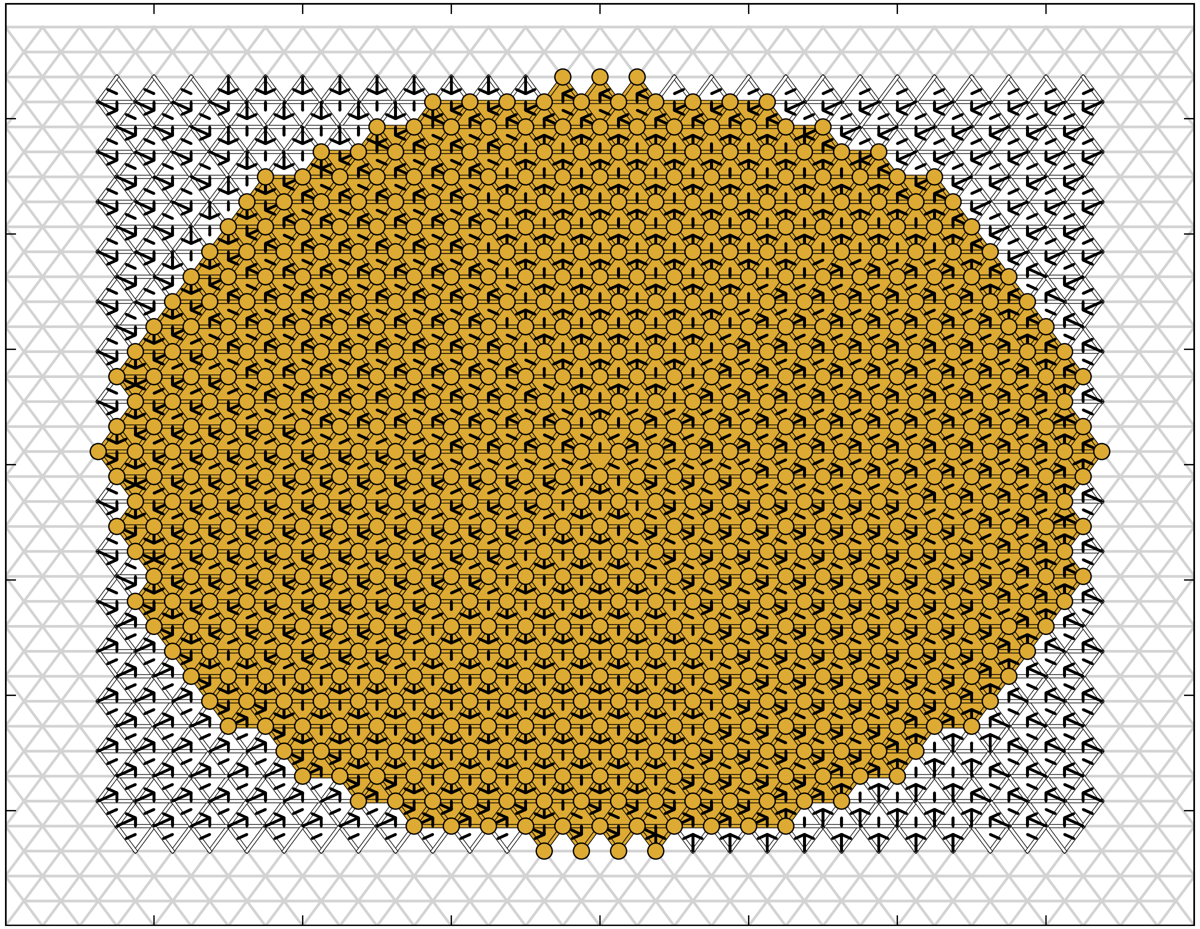

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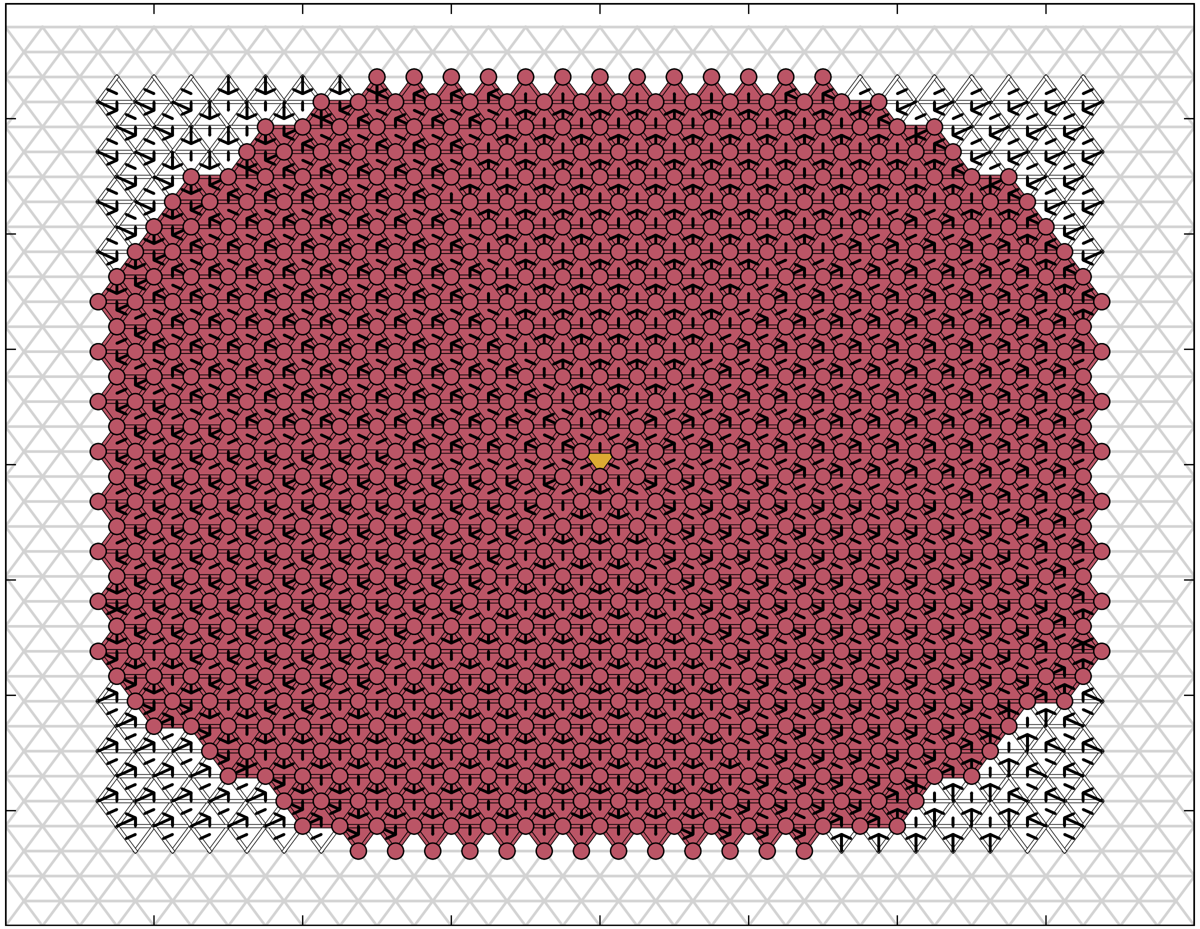

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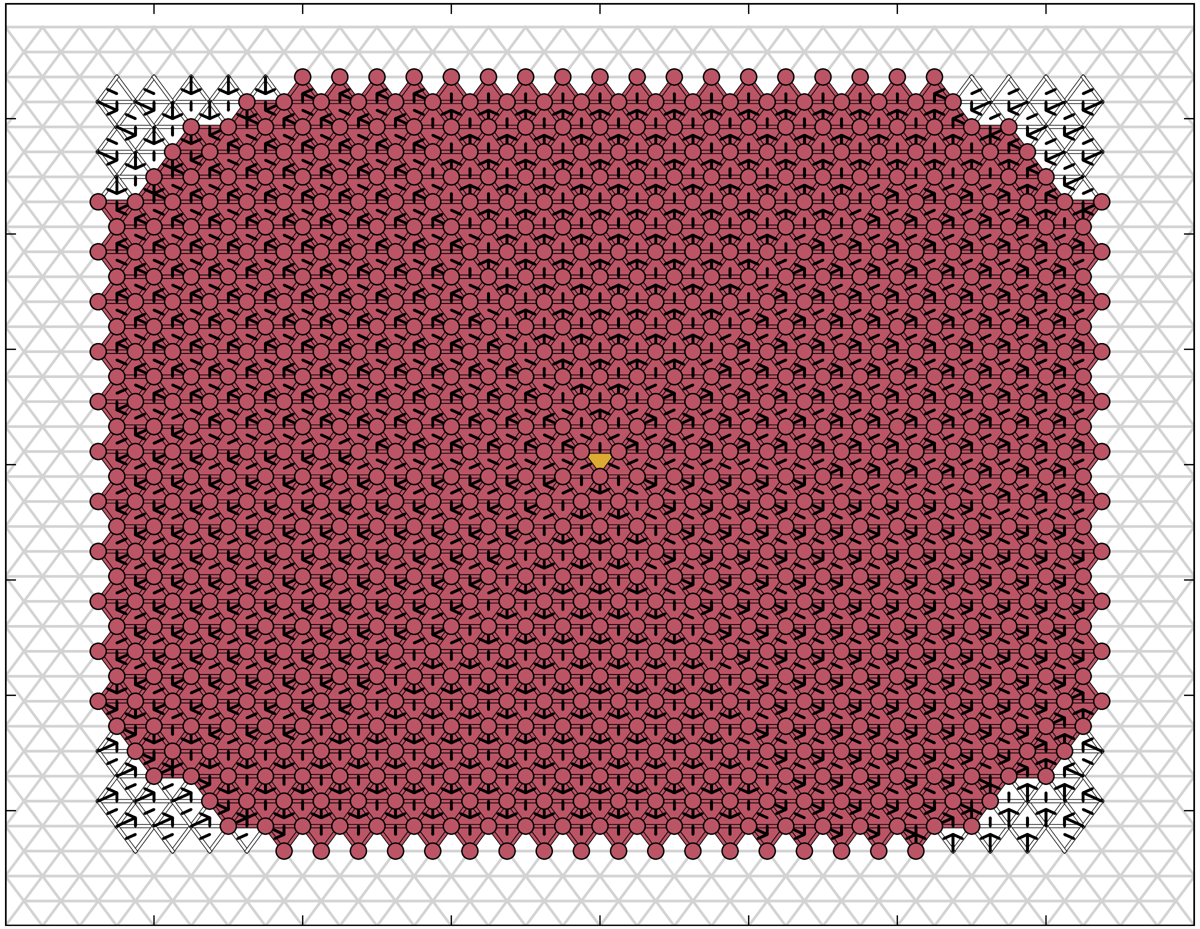


# Motivating Example: Hopf Bifurcation

Note: attractor from $\lambda = -\infty$ to $\lambda = 16$

Can we use persistence to capture this, or a related feature?

**Yes**, using a special type of index pair

# Conley Index Persistence

First attempt: for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute an isolated invariant set, $A_1, A_2, \ldots, A_n$ and corresponding index pairs.

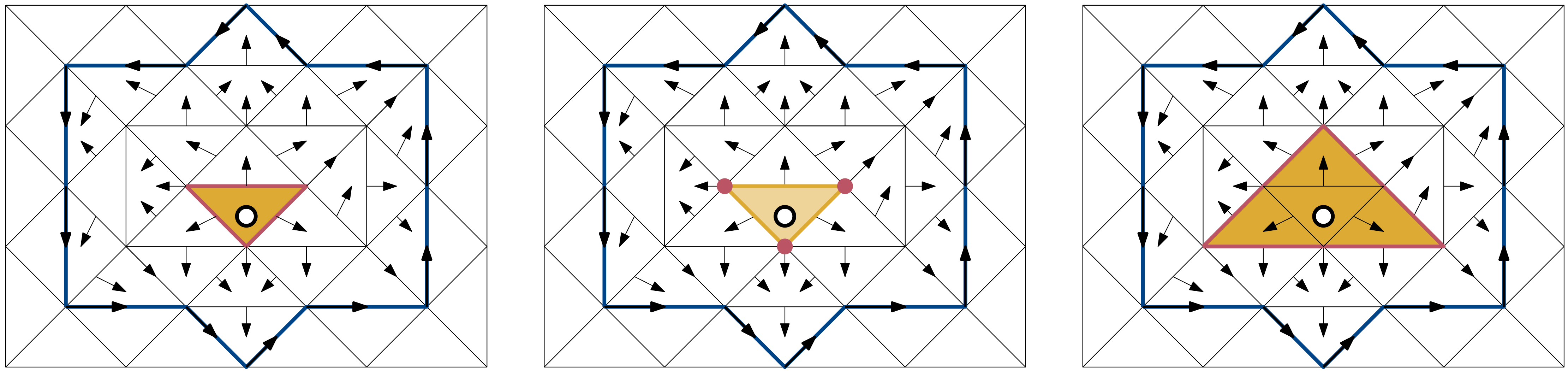
$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

Gives a relative zigzag filtration:

$$\ldots \subseteq (\mathsf{cl}(A_i), \mathsf{mo}(A_i)) \supseteq (\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1})) \subseteq (\mathsf{cl}(A_{i+1}), \mathsf{mo}(A_{i+1})) \supseteq \ldots$$

Problem: $(\mathsf{cl}(A_i) \cap \mathsf{cl}(A_{i+1}), \mathsf{mo}(A_i) \cap \mathsf{mo}(A_{i+1}))$ generally not an index pair.

# Intersection Example

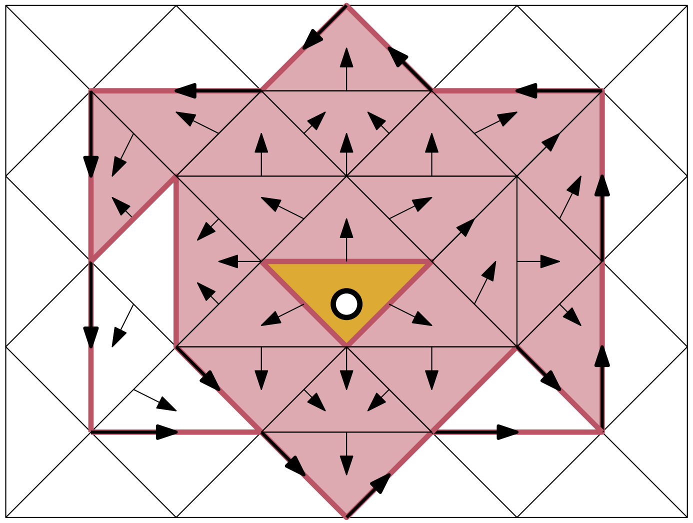


# Index Pairs in an Isolating Neighborhood

Let $E \subset P \subseteq N$ for closed $P$, $E$, $N$, and $A \subseteq N$. If:

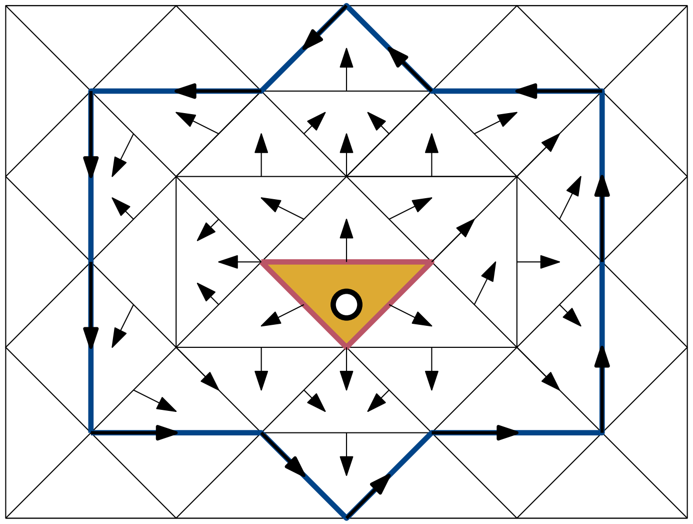
1. $F_{\mathcal{V}}(P) \cap N \subseteq P$,
2. $F_{\mathcal{V}}(E) \cap N \subseteq E$,
3. $F_{\mathcal{V}}(P \setminus E) \subseteq N$, and
4. $A = \mathrm{Inv}(P \setminus E)$

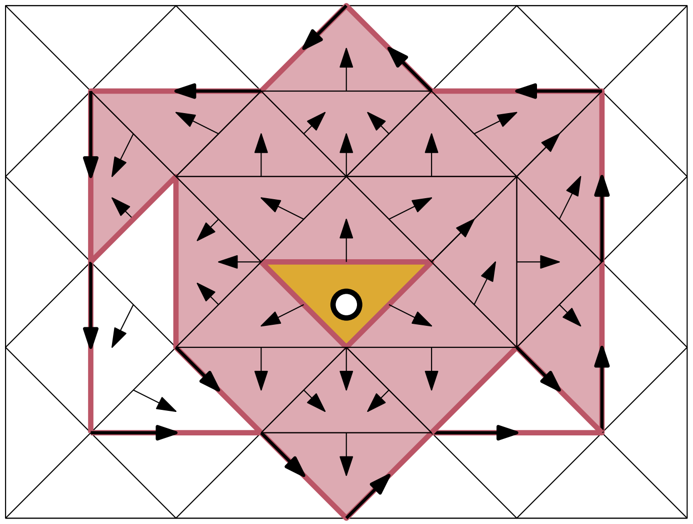
then $(P, E)$ is an index pair in $N$.



# Push Forward

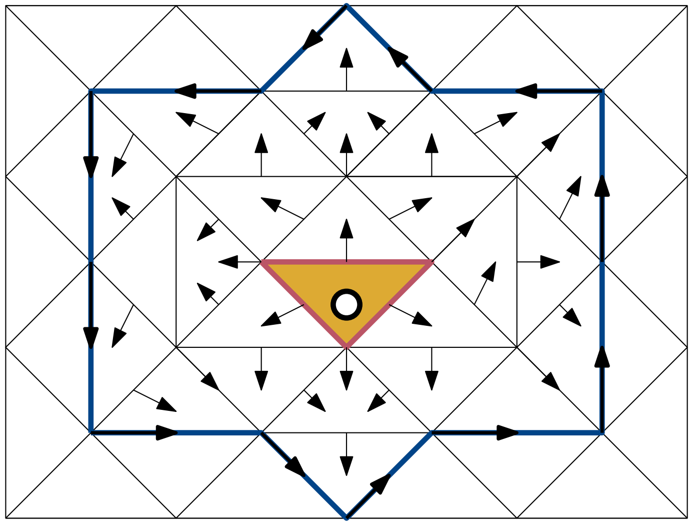
Let $A \subseteq K$ denote an arbitrary set in some closed $N$. Then the push forward of $A$ in $N$ is $A$ together with all simplices in $N$ which are reachable from paths originating in $A$ and contained in $N$.

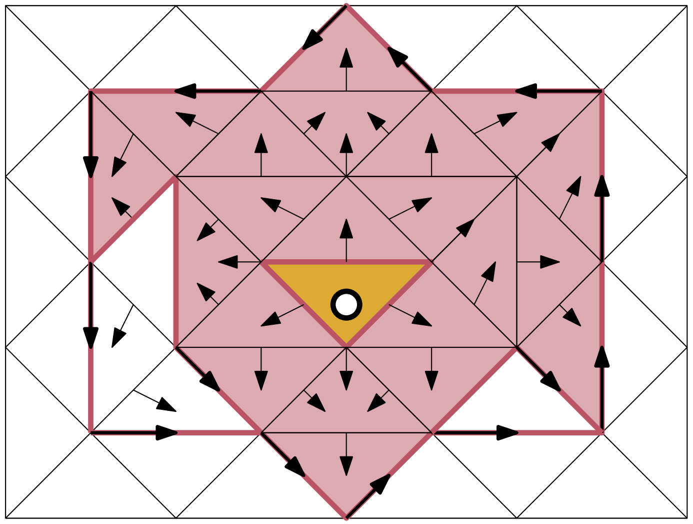




# Finding Index Pairs in N

**Theorem:** The push forward in $N$ of an index pair is an index pair in $N$





# Index Pairs in an Isolating Neighborhood

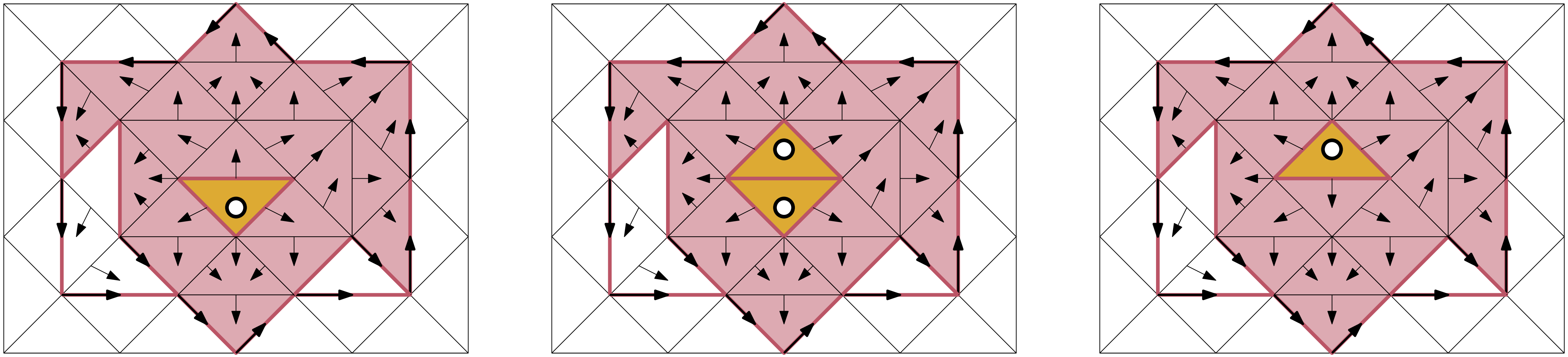
**Theorem:** Index Pairs in $N$ are index pairs.

**Definition:** Let $\mathcal{V}_1, \mathcal{V}_2$ denote multivector fields over $K$. The intersection of multivector fields is given by

$$\mathcal{V}_1 \overline{\cap} \mathcal{V}_2 = \{V_1 \cap V_2 \mid V_1 \in \mathcal{V}_1,\ V_2 \in \mathcal{V}_2\}$$

**Theorem:** Let $(P_1, E_1), (P_2, E_2)$ denote index pairs in $N$ under $\mathcal{V}_1, \mathcal{V}_2$. The pair $(P_1 \cap P_2, E_1 \cap E_2)$ is an index pair in $N$ under $\mathcal{V}_1 \overline{\cap} \mathcal{V}_2$ for $\mathrm{Inv}((P_1 \cap P_2) \setminus (E_1 \cap E_2))$.

# Intersection Example



All simplices in N, Yellow union
Red is P, and Red is E

# Conley Index Persistence: New Strategy

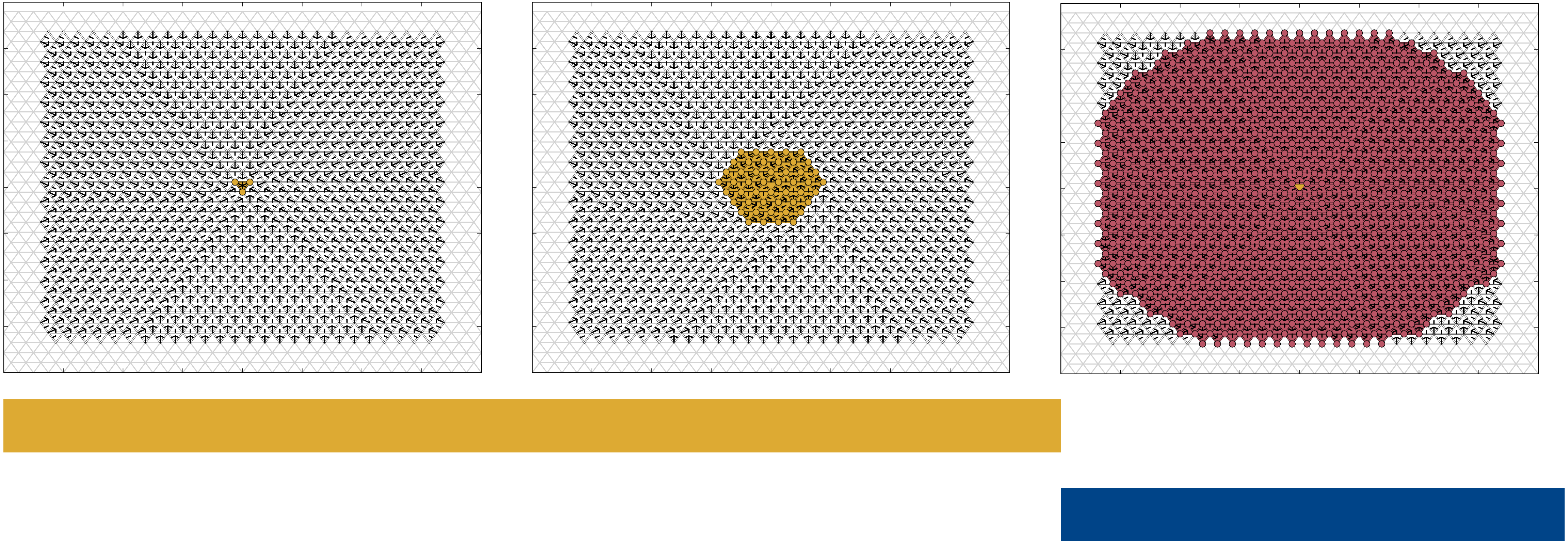
Fix $N$, and for each $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$, compute the maximal invariant set in $N$, denoted $A_1, A_2, \ldots, A_n$, and corresponding index pairs.

$$(\mathsf{cl}(A_1), \mathsf{mo}(A_1)), (\mathsf{cl}(A_2), \mathsf{mo}(A_2)), \ldots, (\mathsf{cl}(A_n), \mathsf{mo}(A_n))$$

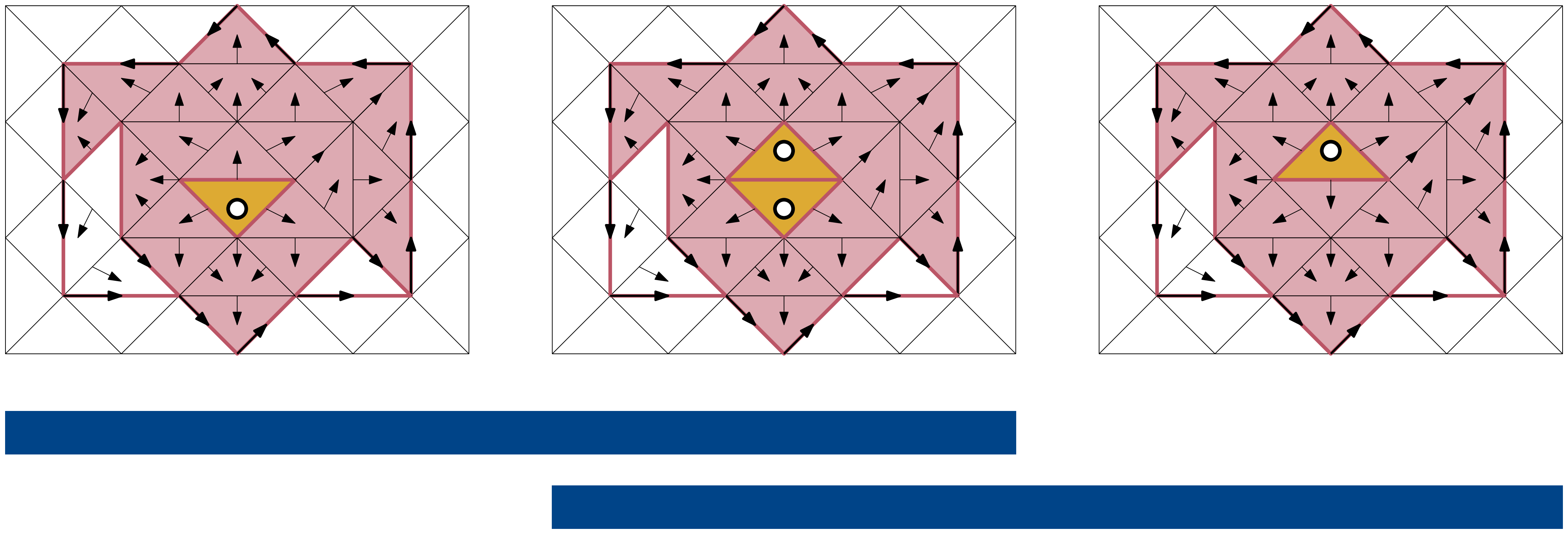
Gives a relative zigzag filtration:

$$(\mathsf{pf}_N(\mathsf{cl}A_i), \mathsf{pf}_N(\mathsf{mo}A_i)) \supseteq (\mathsf{pf}_N(\mathsf{cl}A_i) \cap \mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_i) \cap \mathsf{pf}_N(\mathsf{mo}A_{i+1})) \subseteq (\mathsf{pf}_N(\mathsf{cl}A_{i+1}), \mathsf{pf}_N(\mathsf{mo}A_{i+1}))$$

# Conley Index Persistence

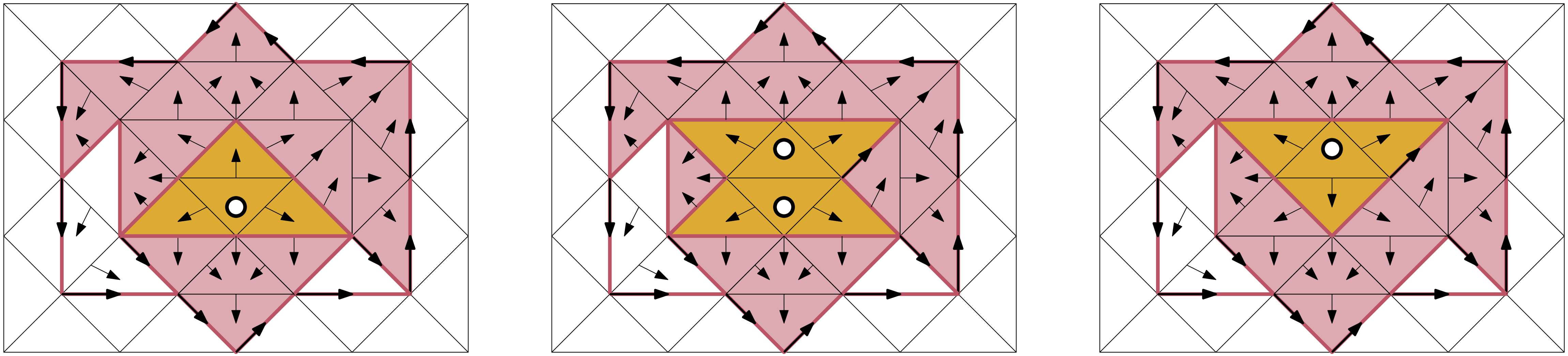


# Problem: Noise Resilience



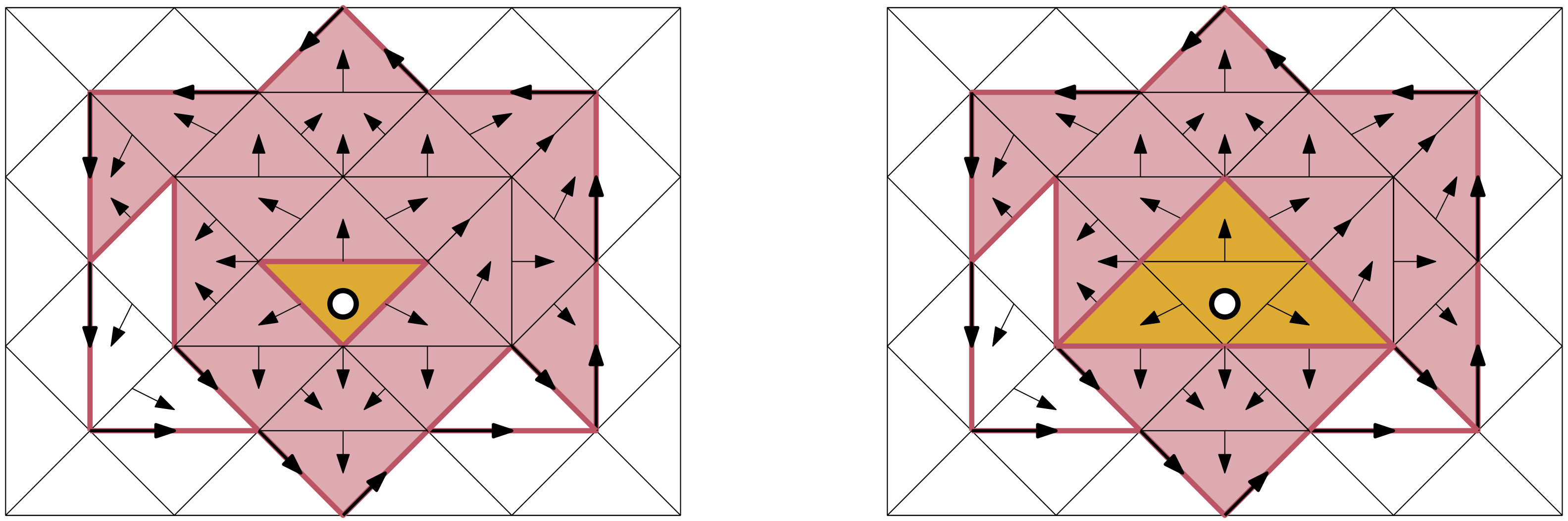
All simplices in N, Yellow union
Red is P, and Red is E

# Solution: Make E Smaller



# Conley Index Persistence

**Proposition:** Let $(P, E)$ denote an index pair for $A$ in $N$. If $V \subseteq E$ is a regular multivector such that $E' := E \setminus V$ is closed, then $(P, E')$ is an index pair in $N$ for $A$.



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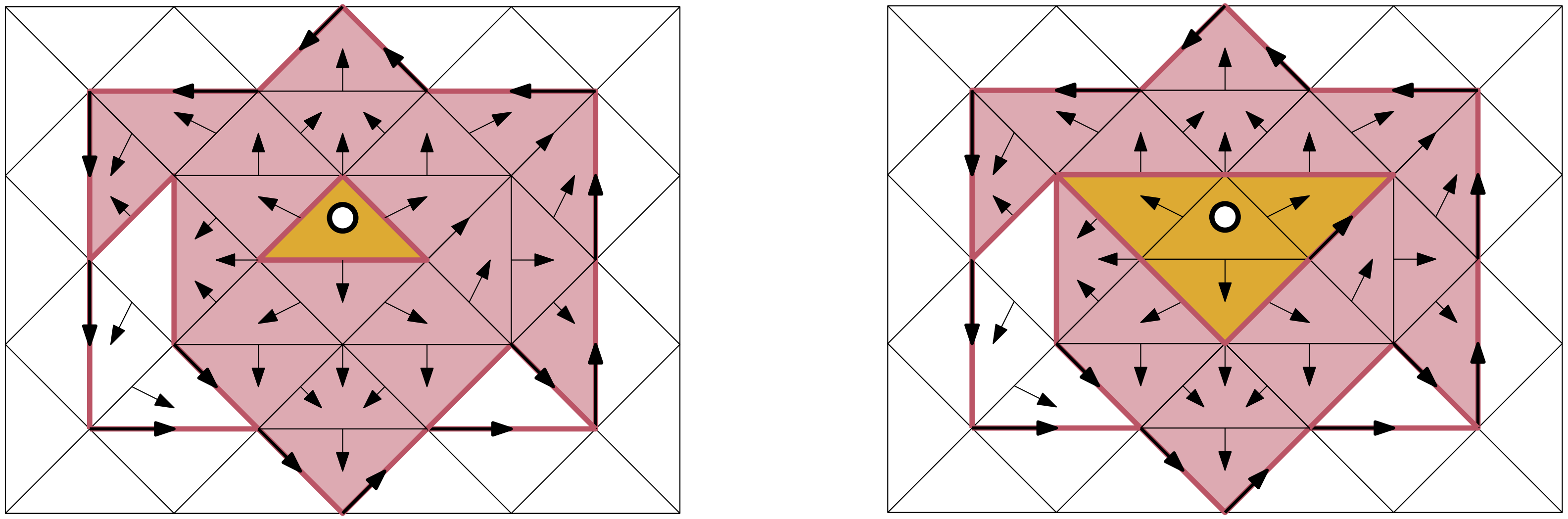
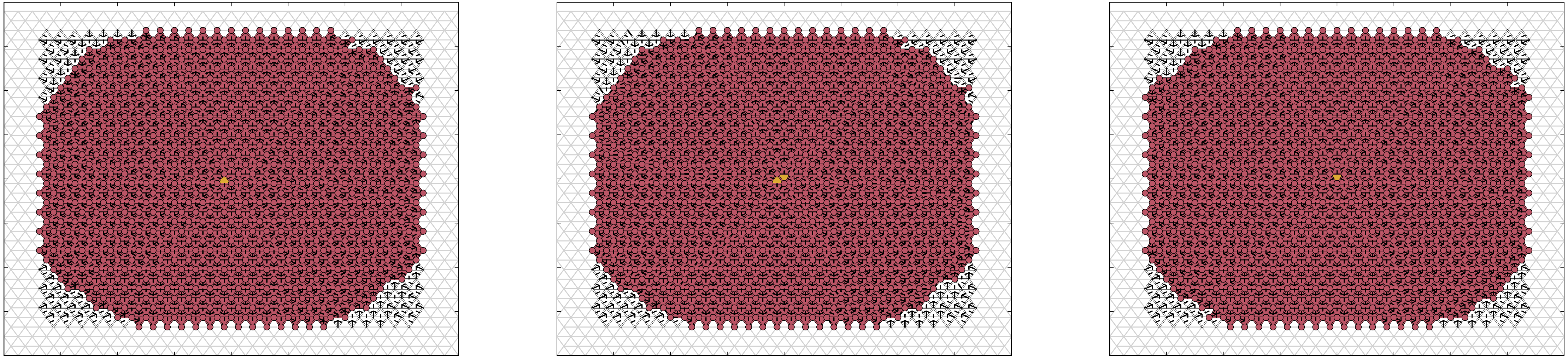


# Multivector Removal Strategy

Remove as many multivectors as possible, up to a fixed distance away from the isolated invariant set.



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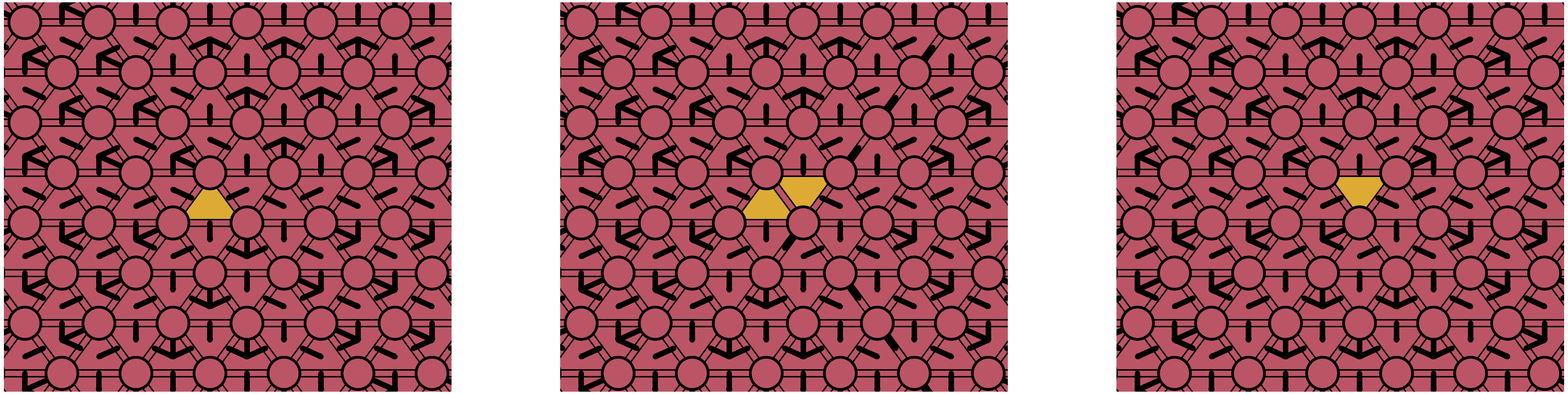

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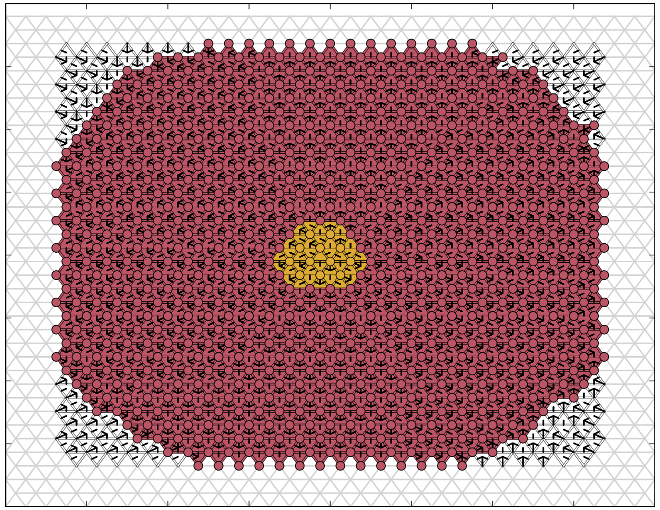
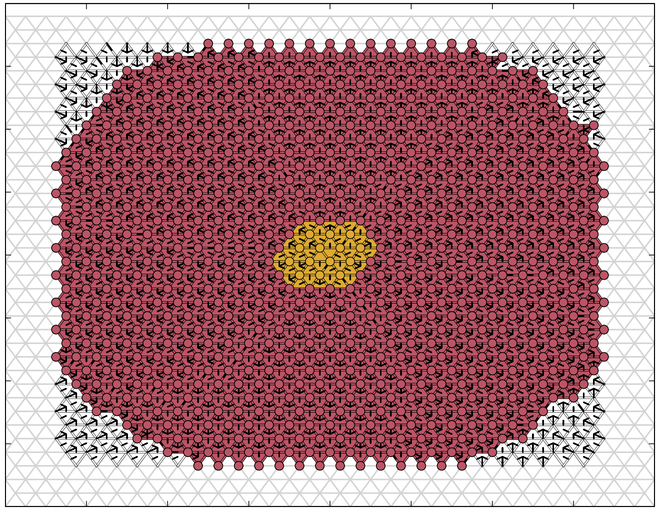
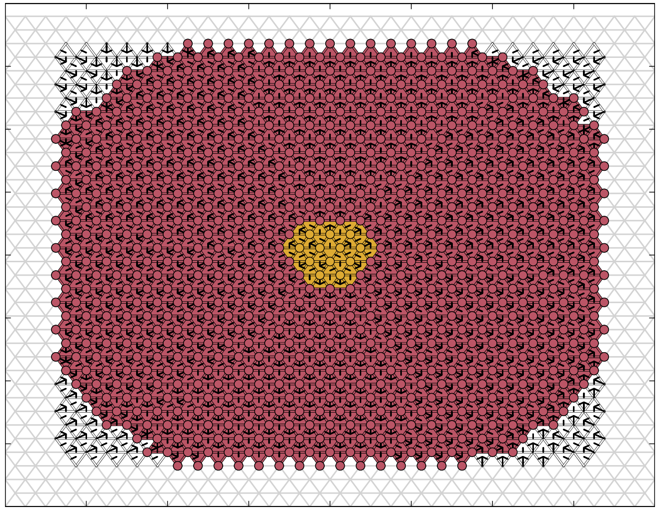




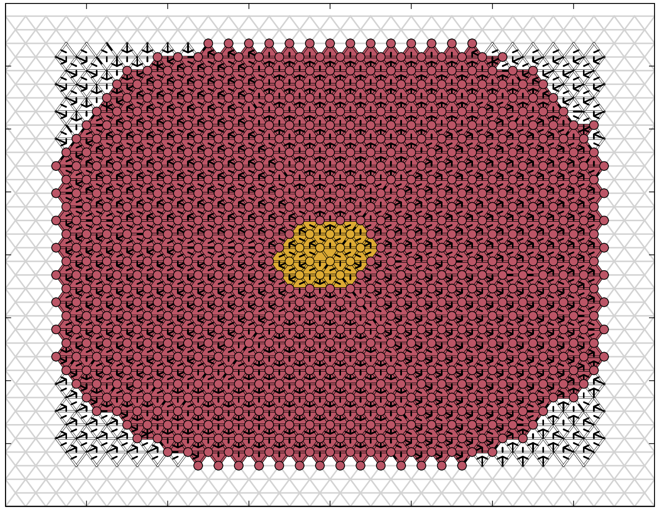


# Algorithm

**MakeNoiseResilient(** P, E, A, $\delta$ ):

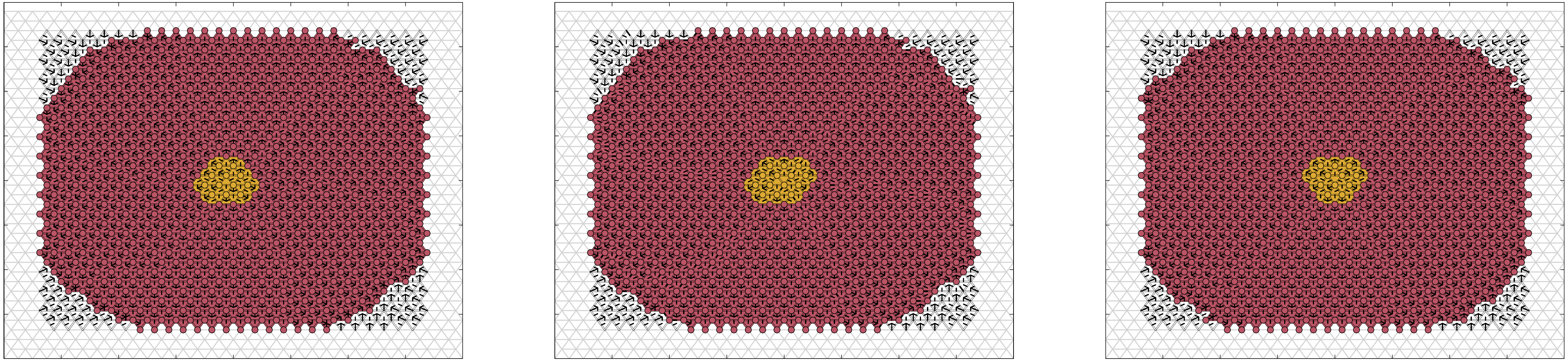
**while** there exists a regular multivector $V \subset E$ such that $E \setminus V$ is closed and $d(V, A) \leq \delta$ :

$$E \leftarrow E \setminus V$$



# Multivector Removal Strategy

**Theorem:** This algorithm outputs index pairs



# Conley Index Persistence: Variable N

Given a “seed” isolated invariant set and isolating neighborhood, can we modify N to track changes to the invariant set across multivector fields?

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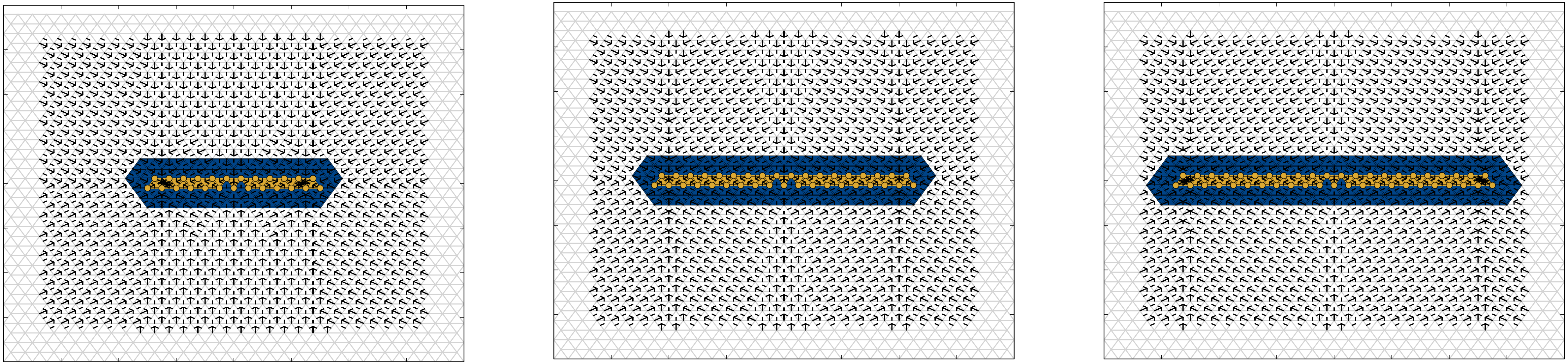
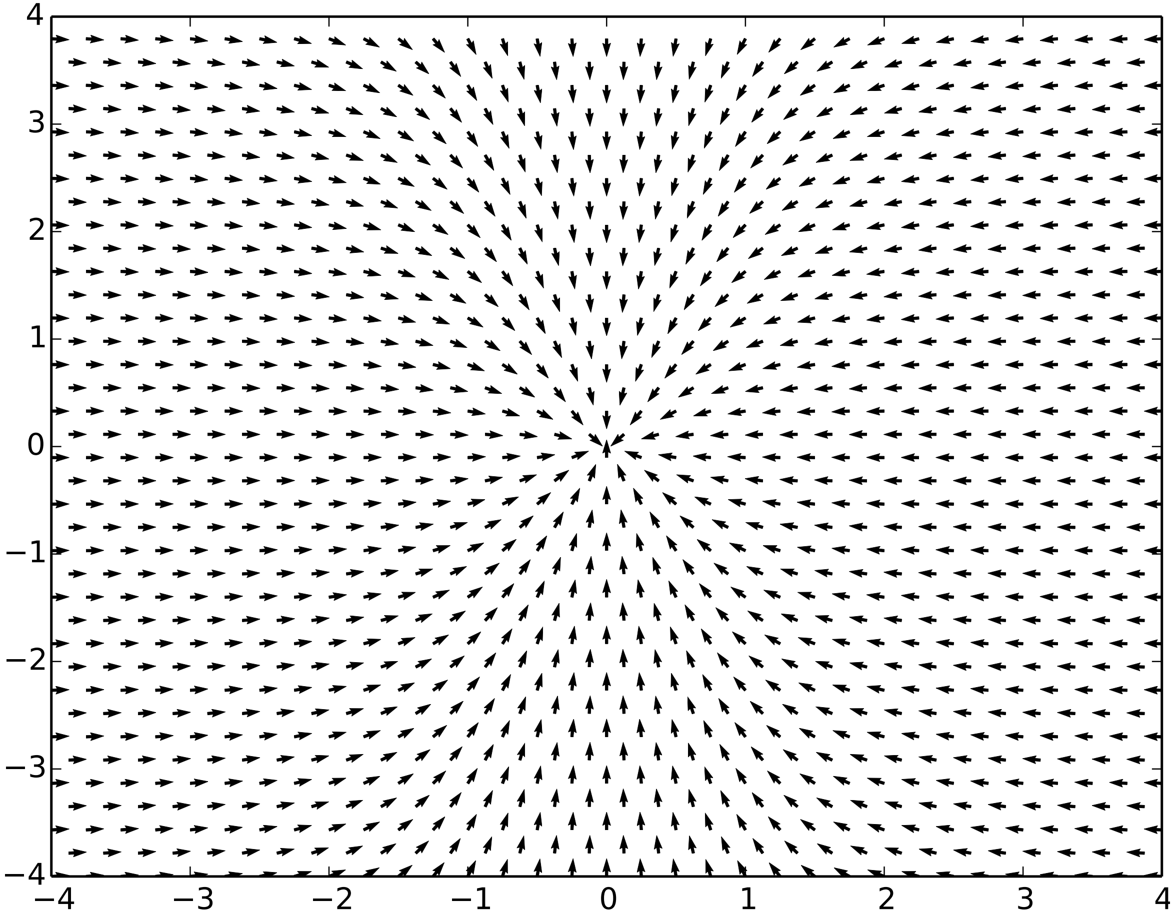


# Pitchfork Bifurcation

$$\frac{dx}{dt} = \lambda x - x^3$$

$$\frac{dy}{dt} = -y$$

# Pitchfork Bifurcation



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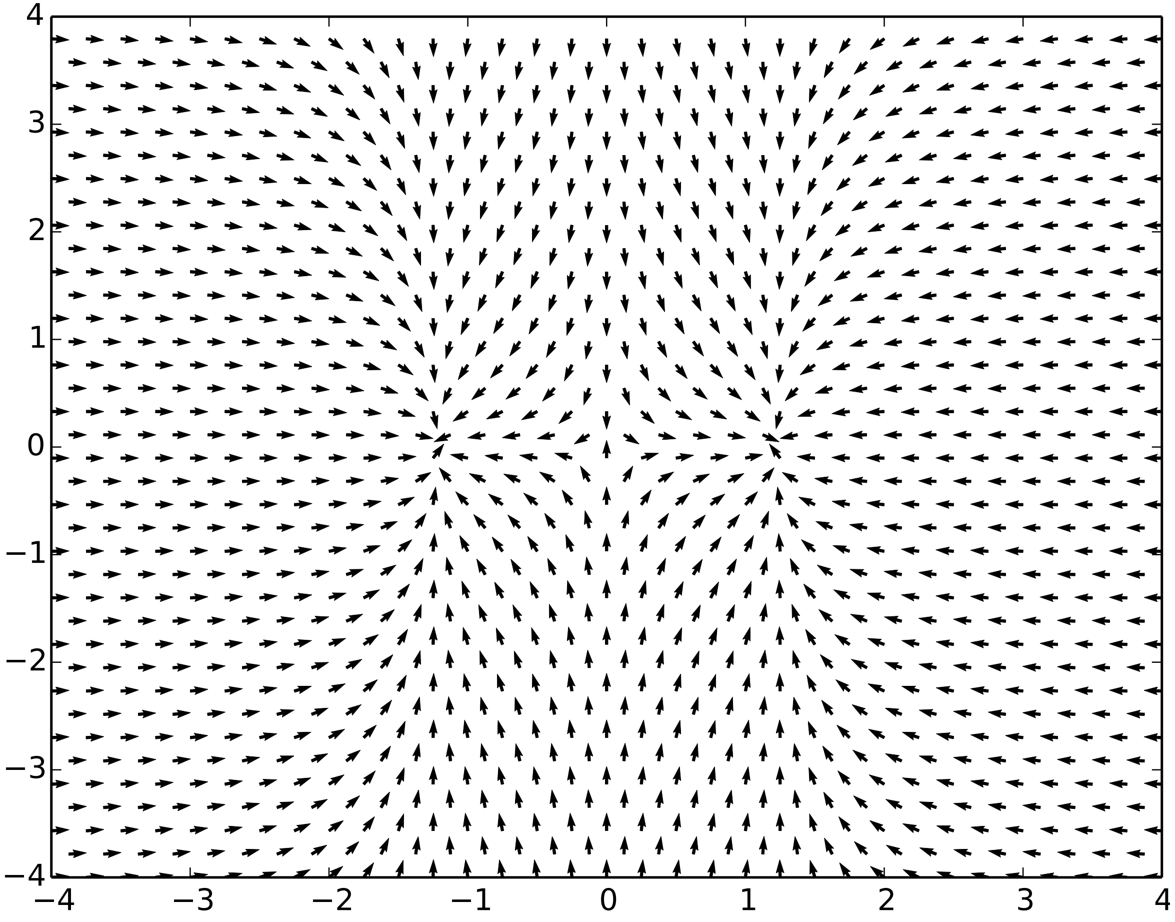

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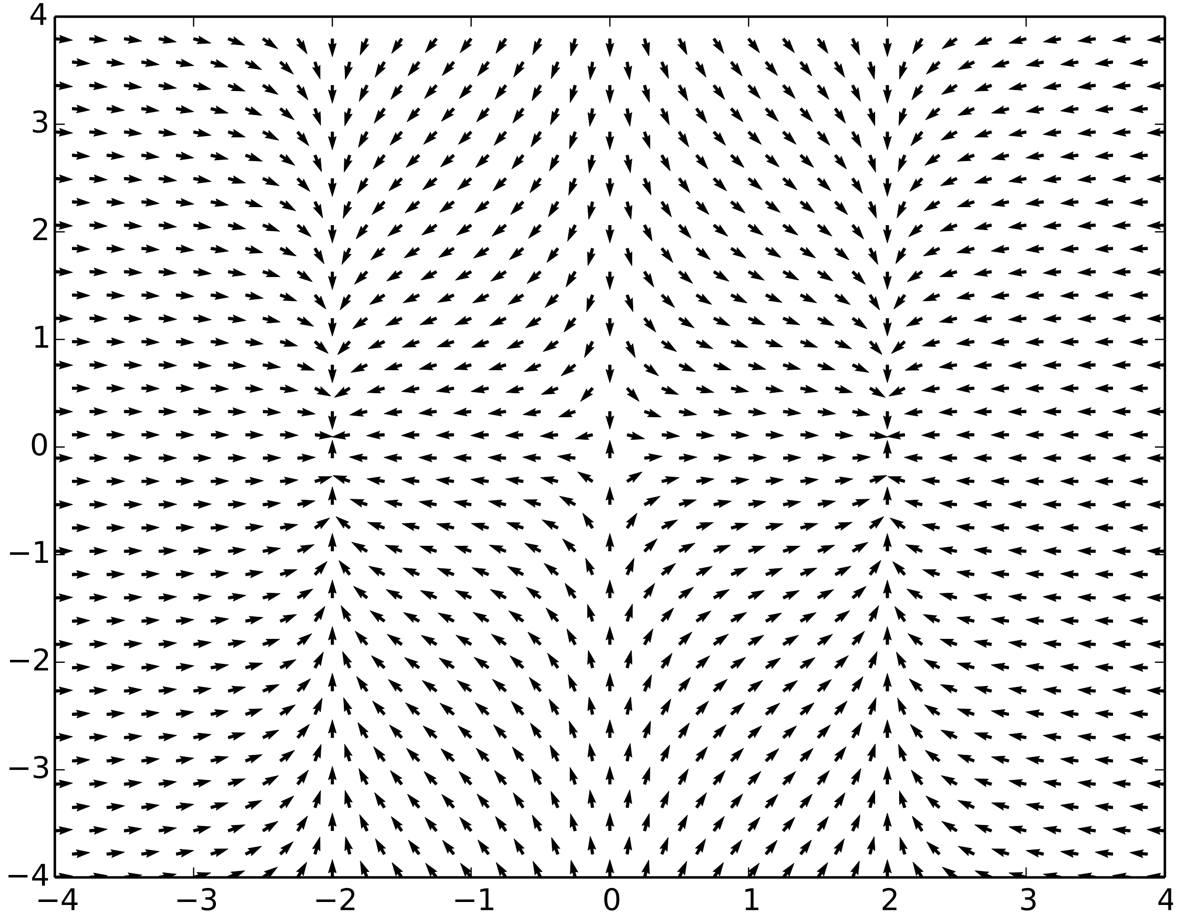

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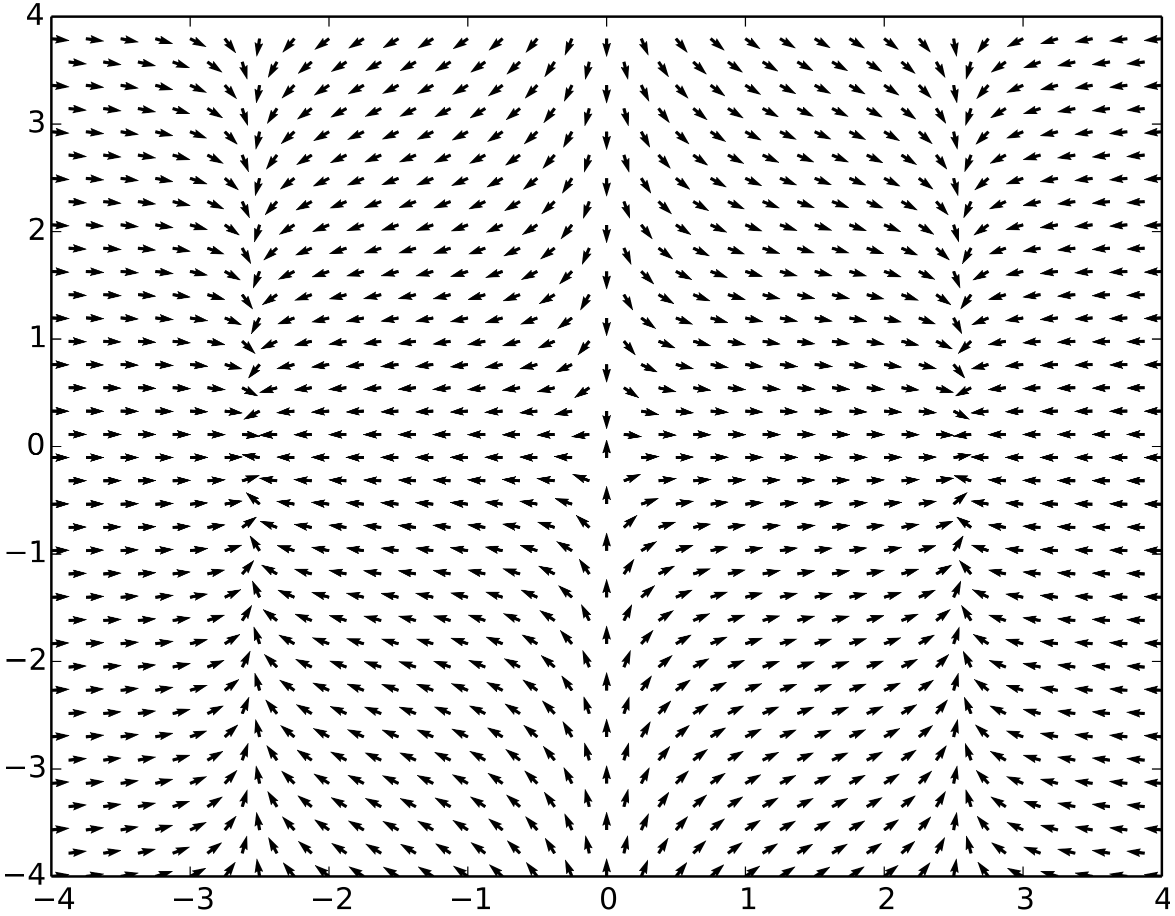

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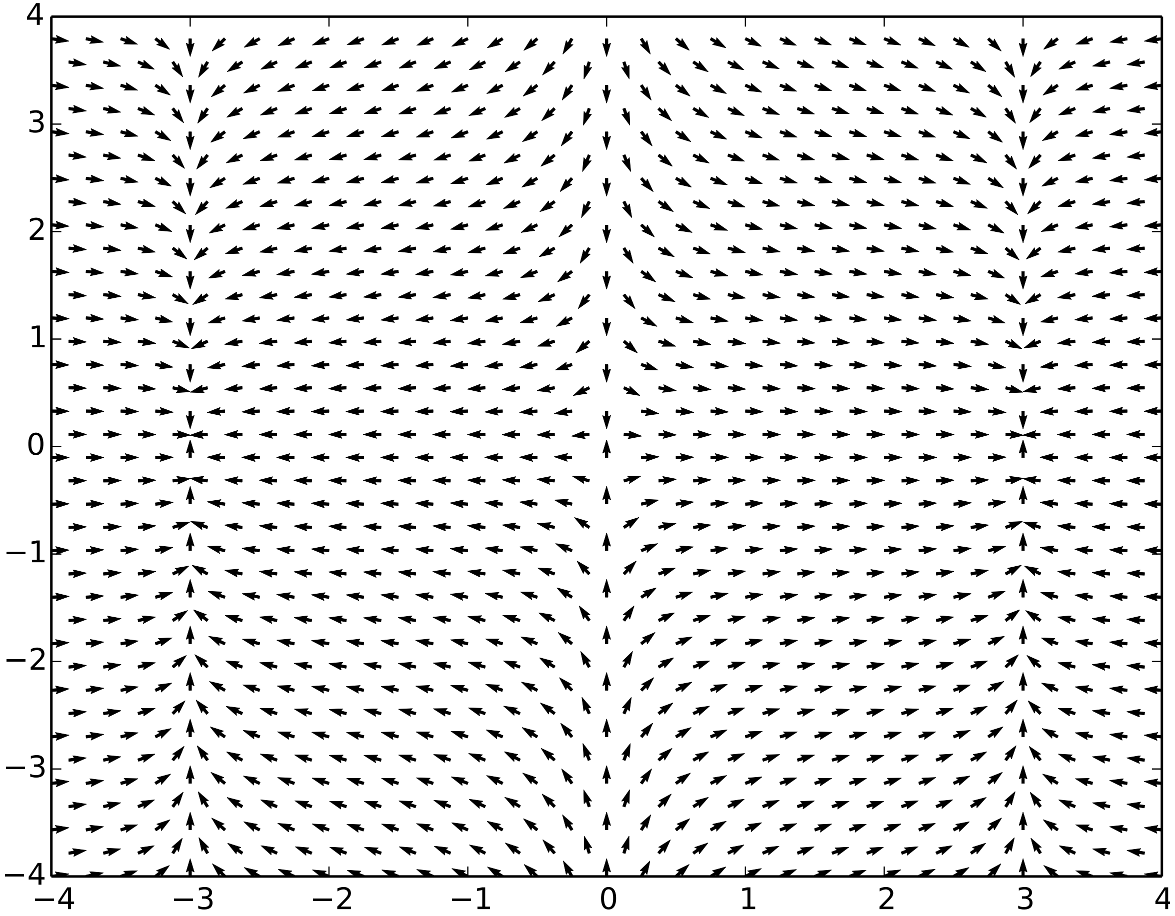

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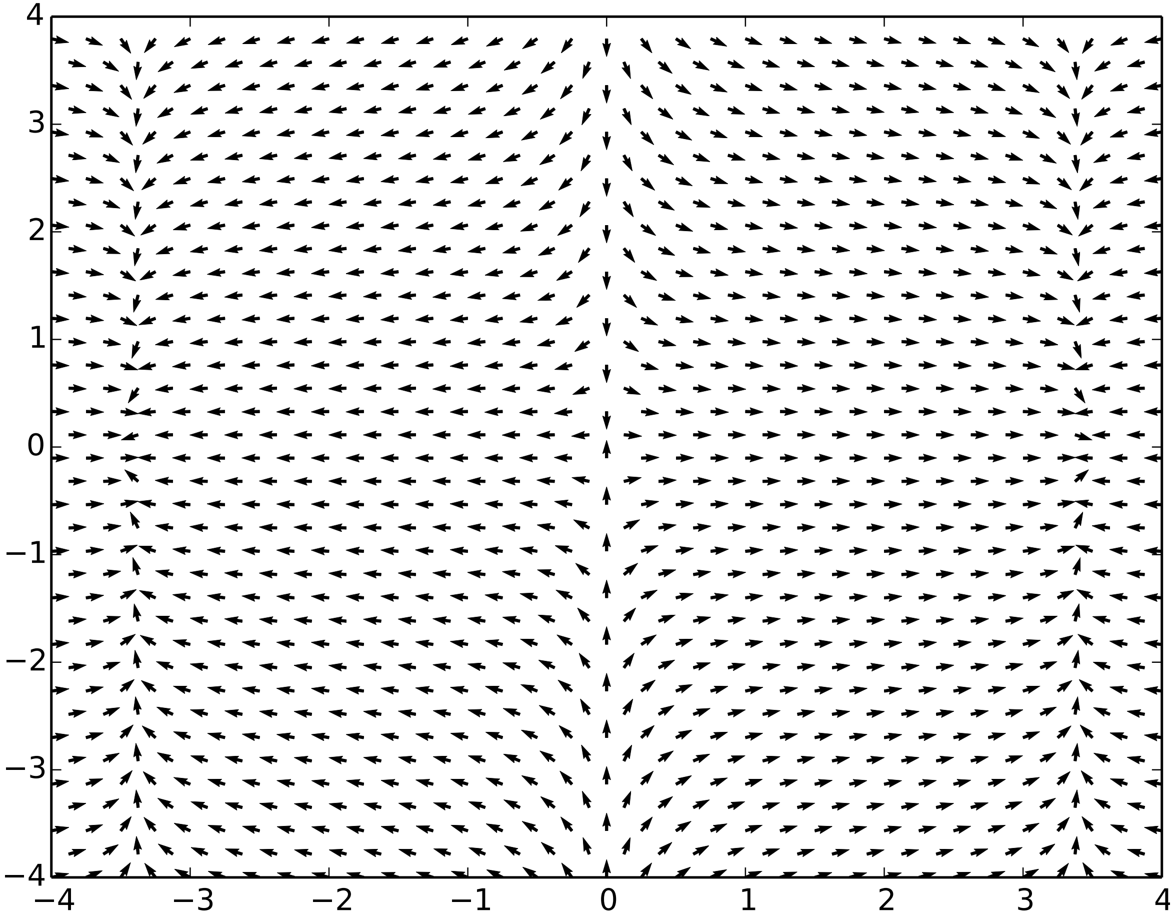

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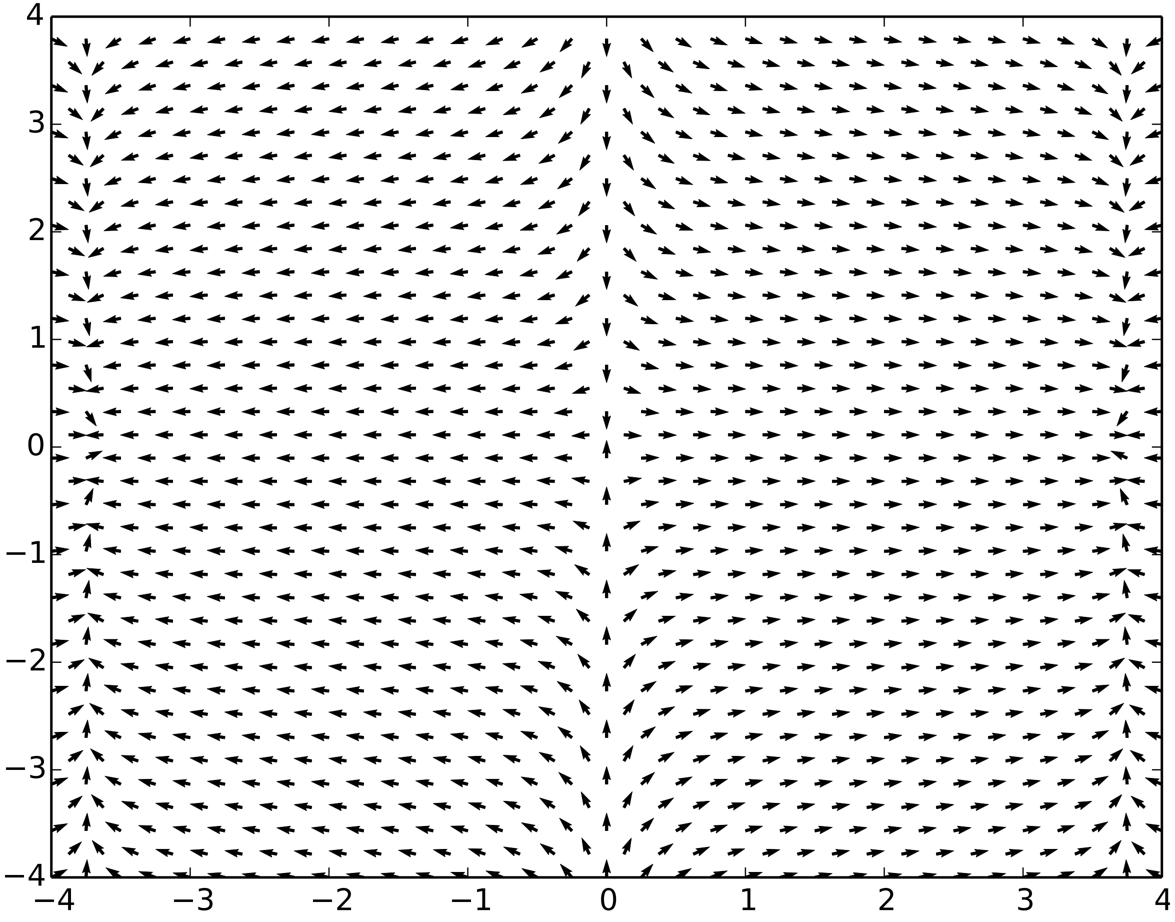

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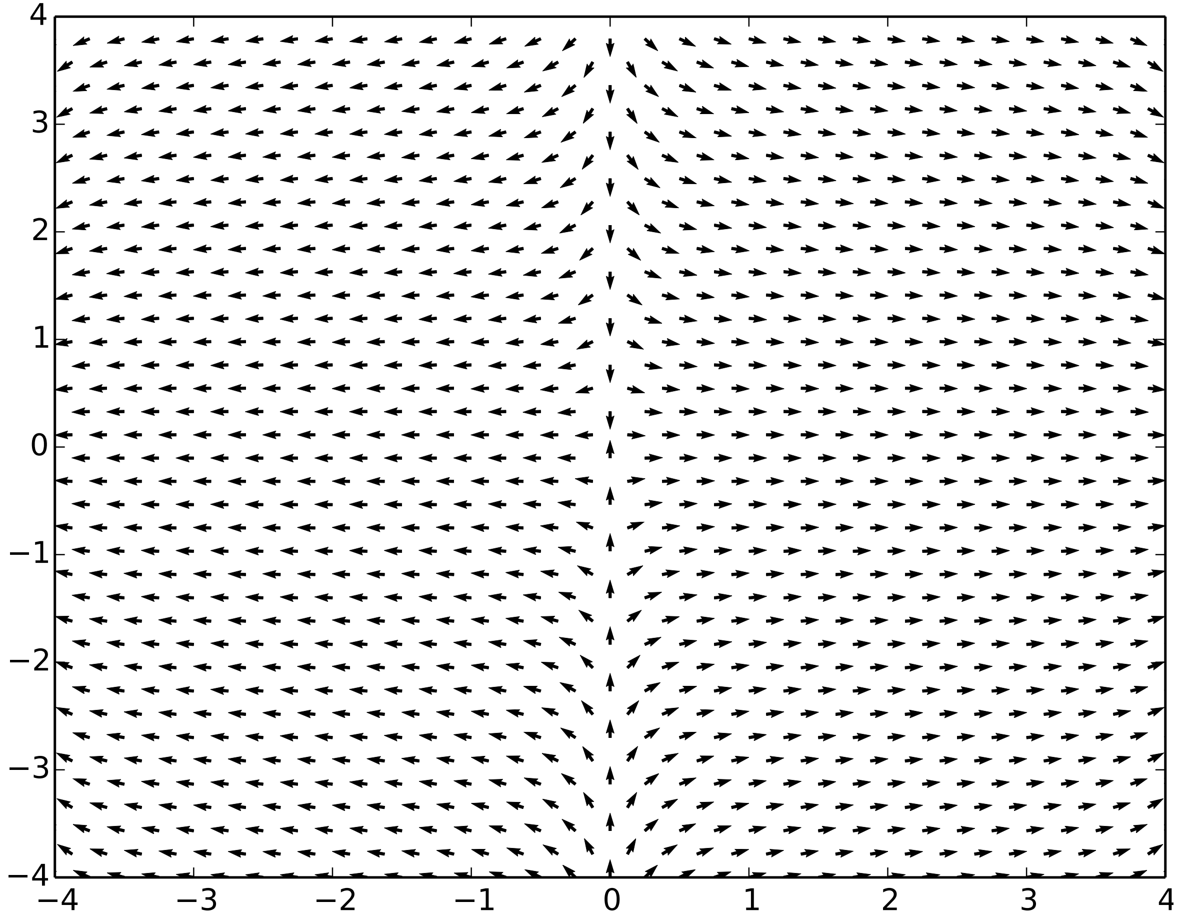

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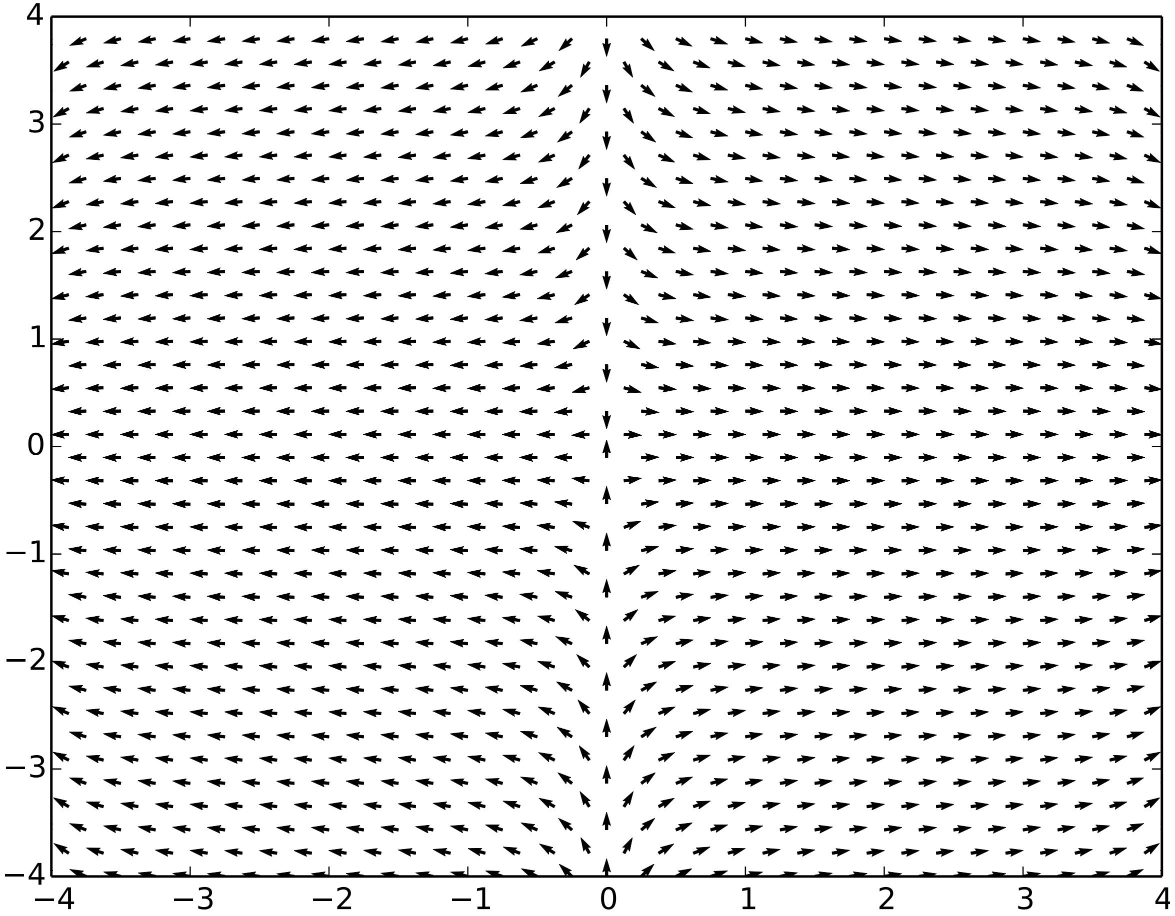

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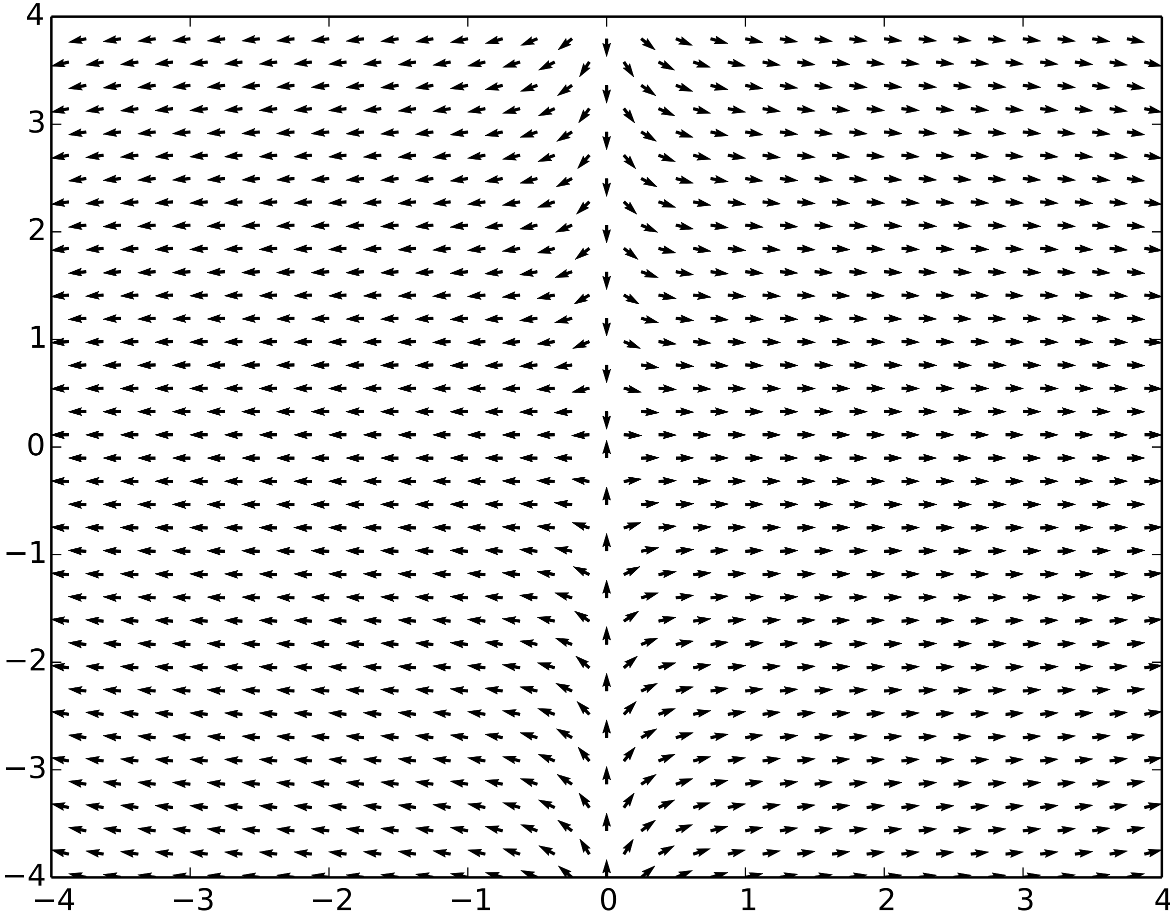
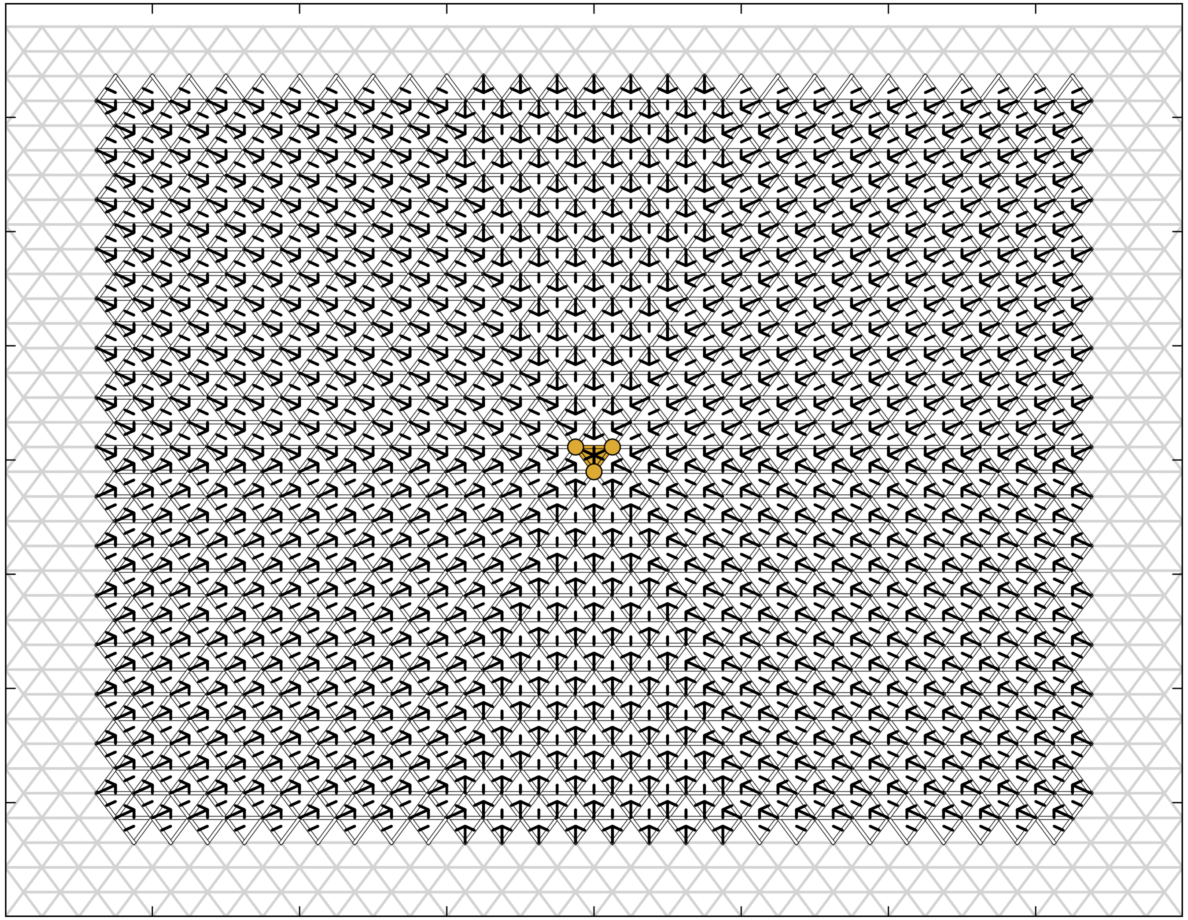


# Pitchfork Bifurcation – Previous Technique



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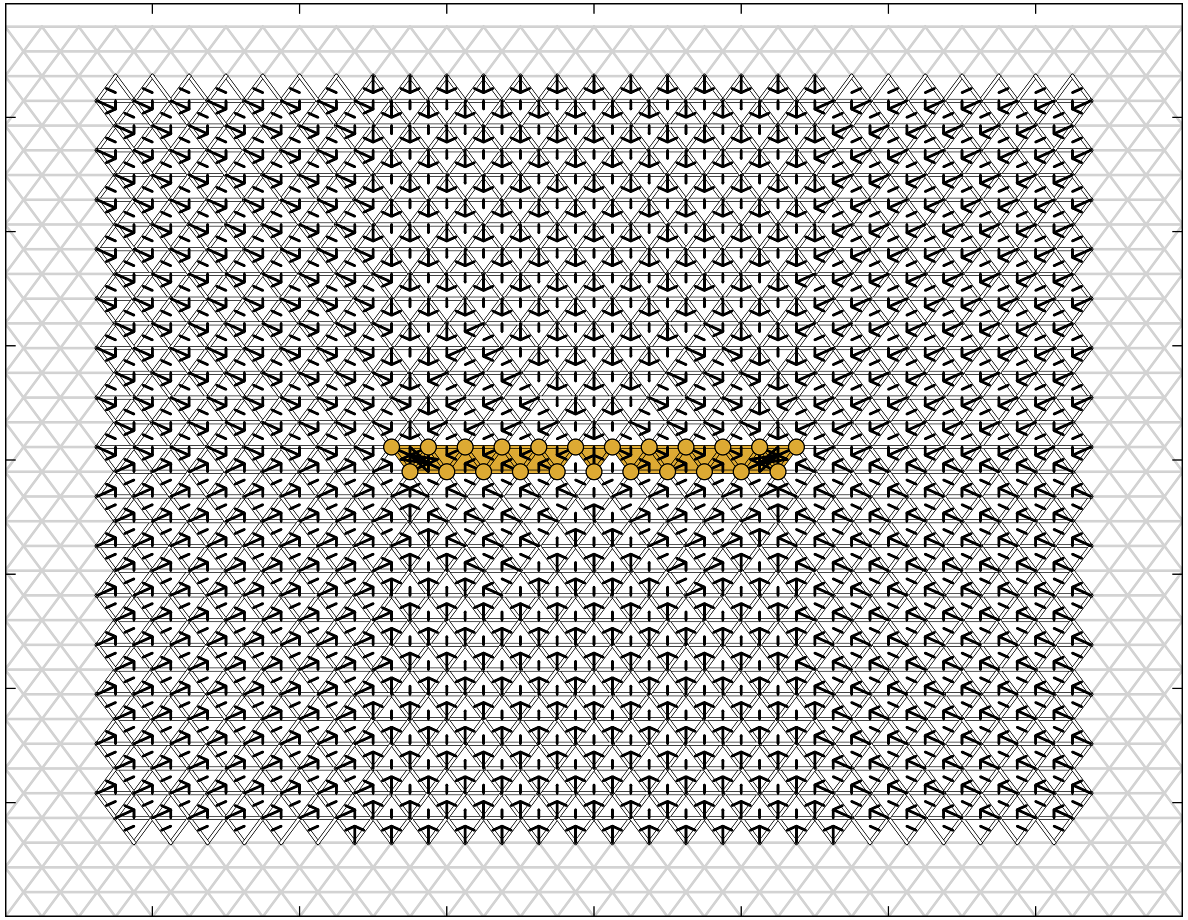

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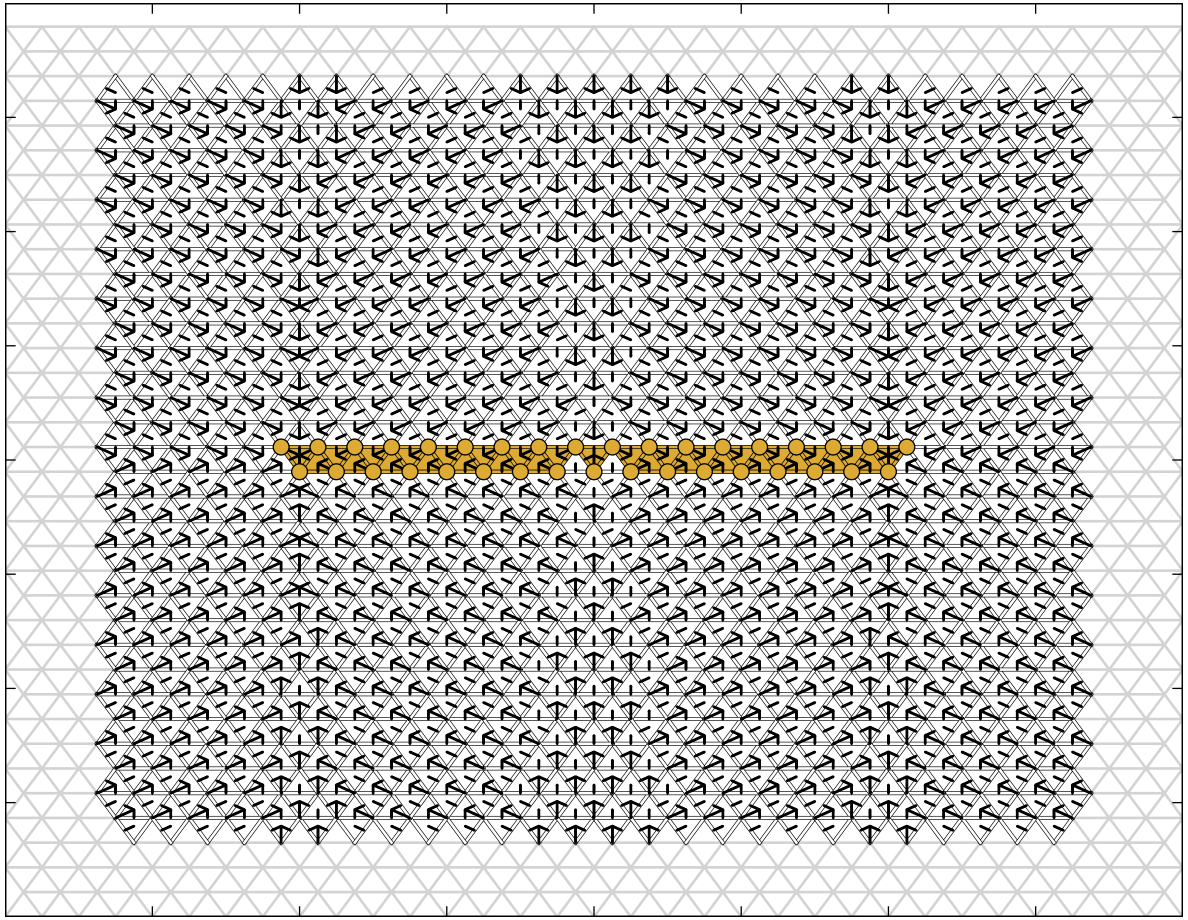

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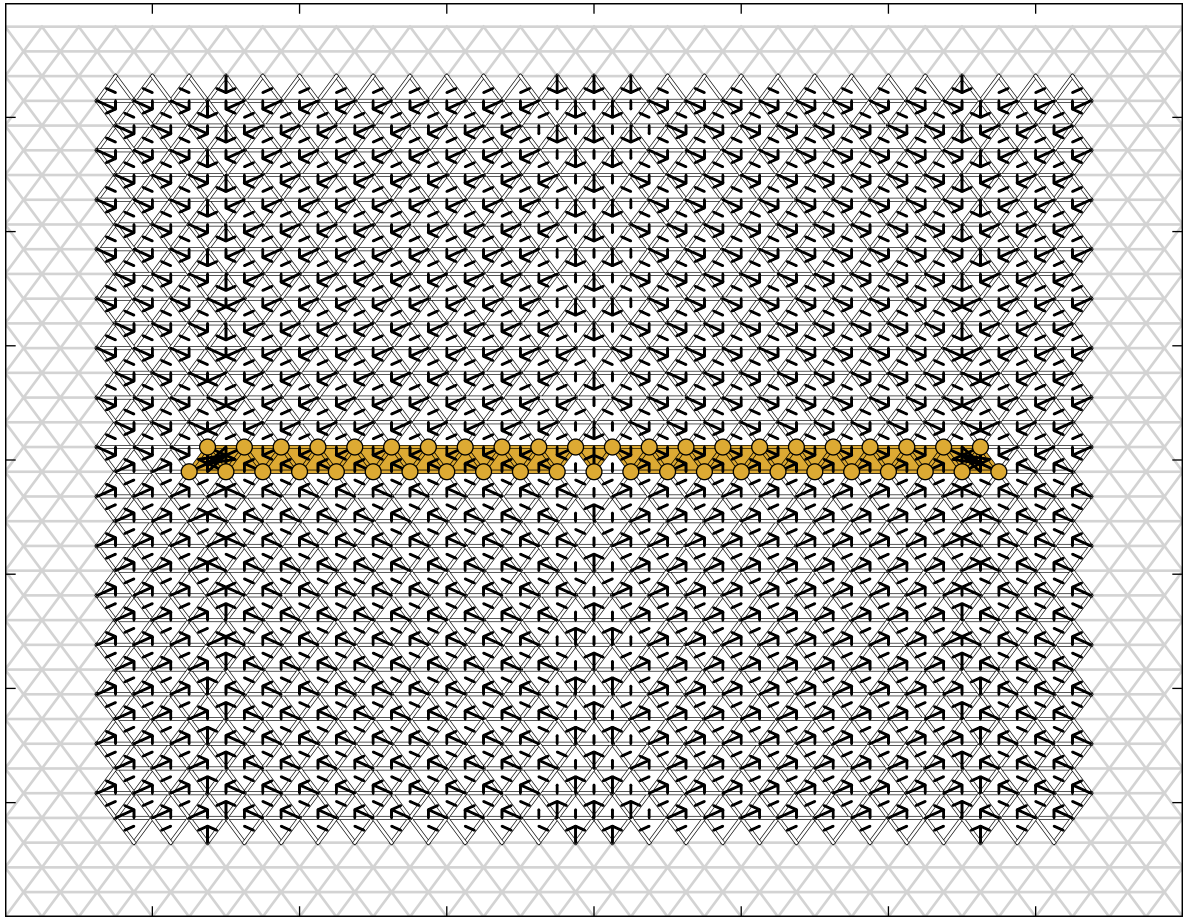

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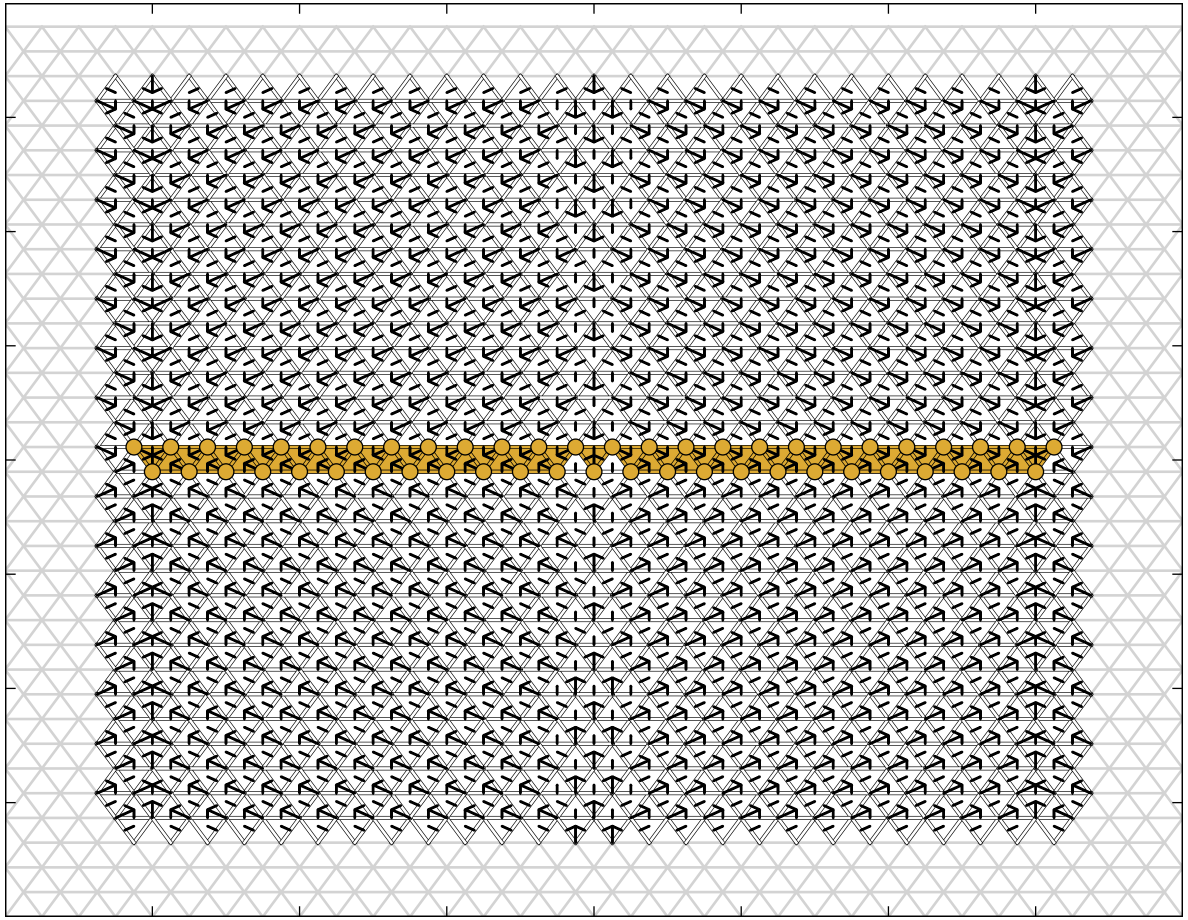

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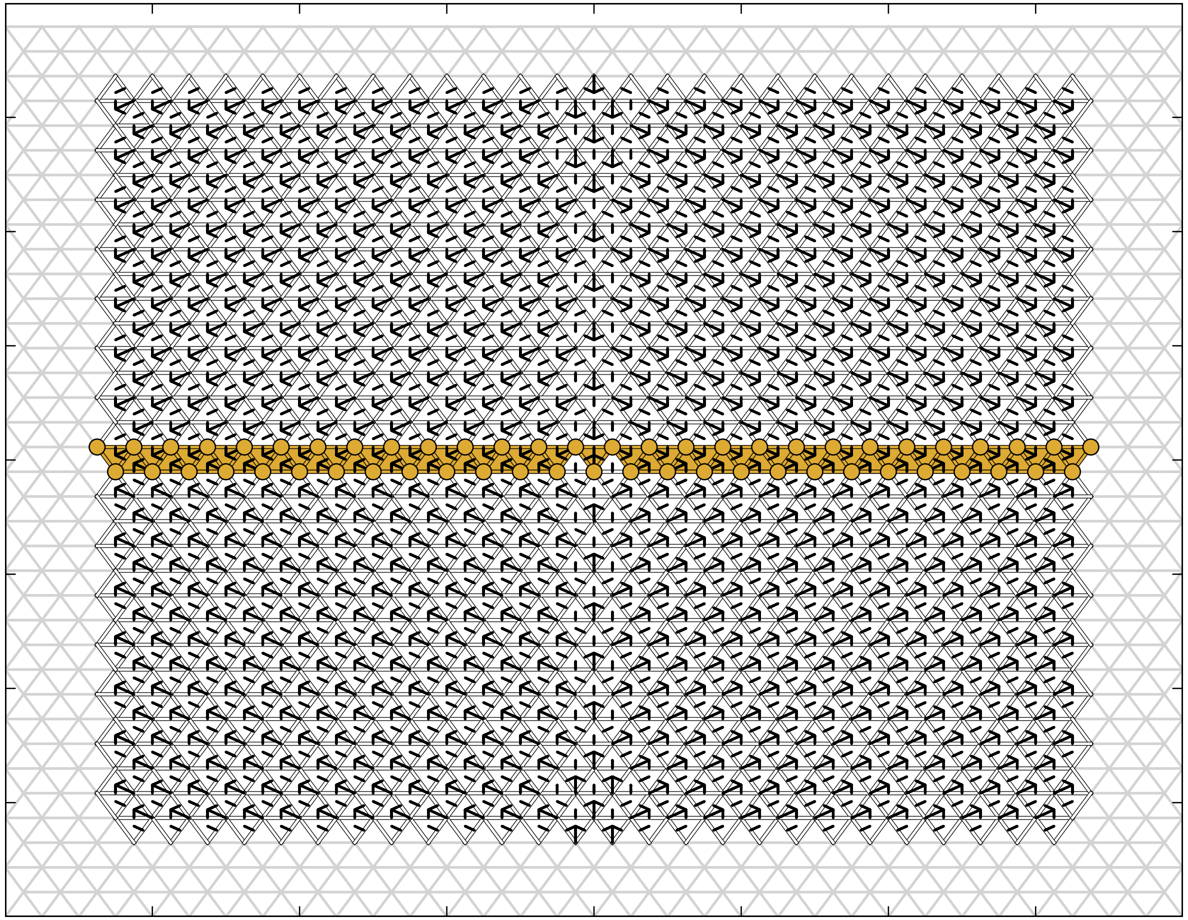

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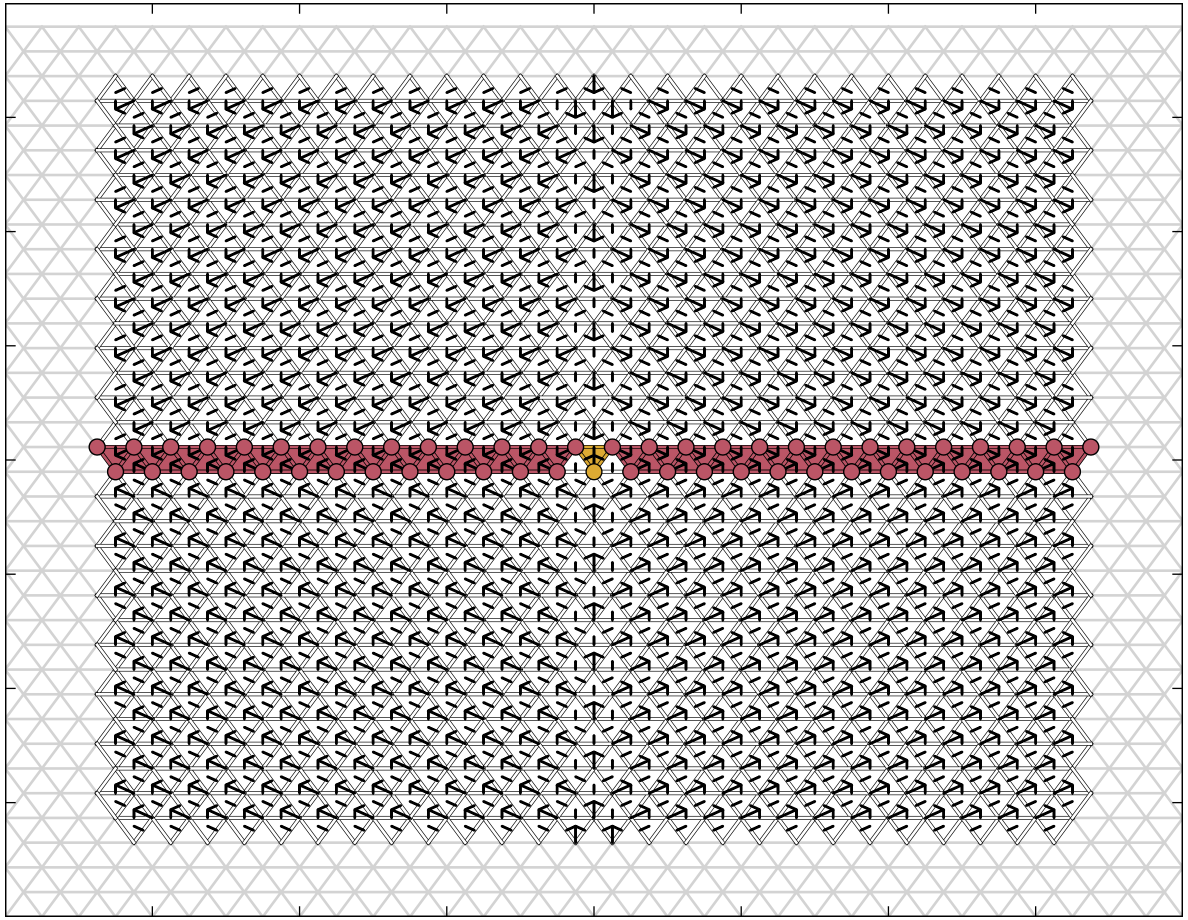

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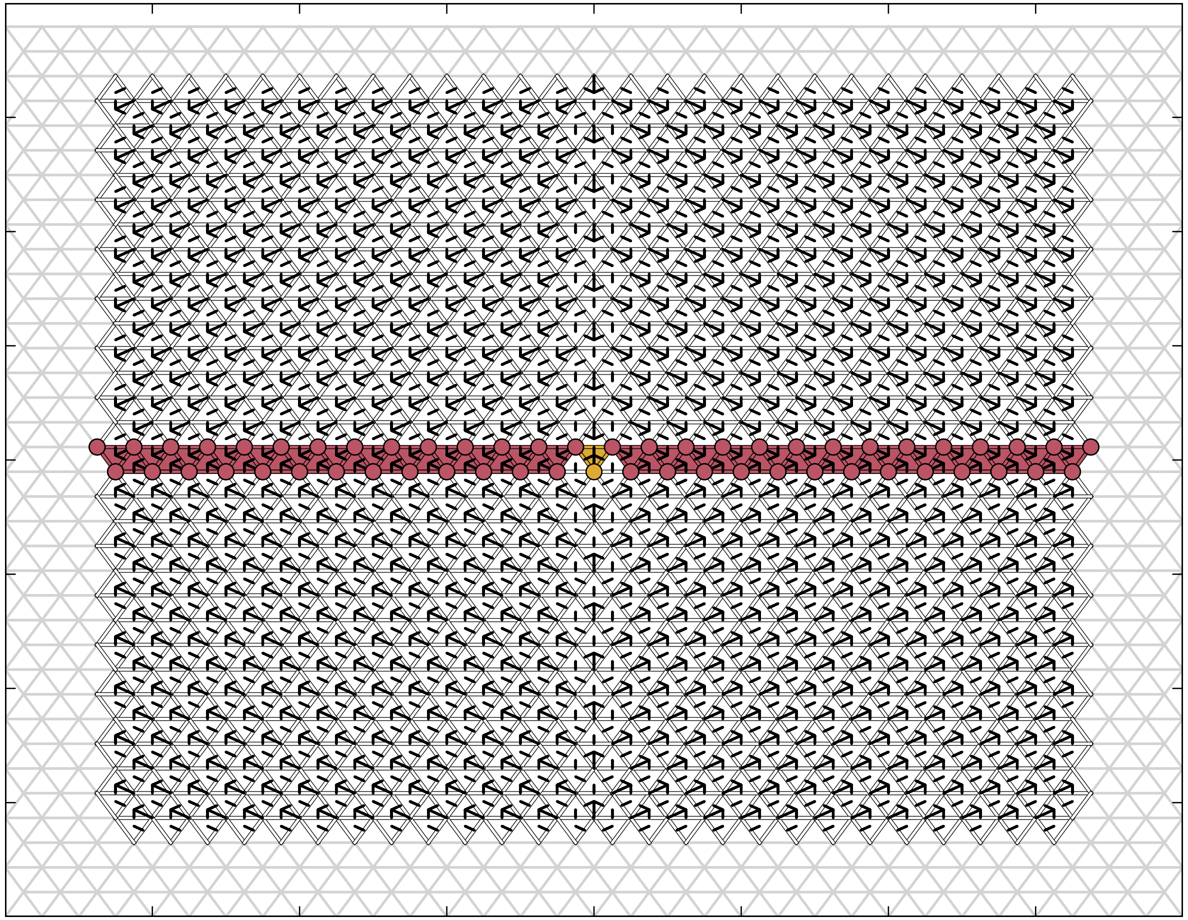

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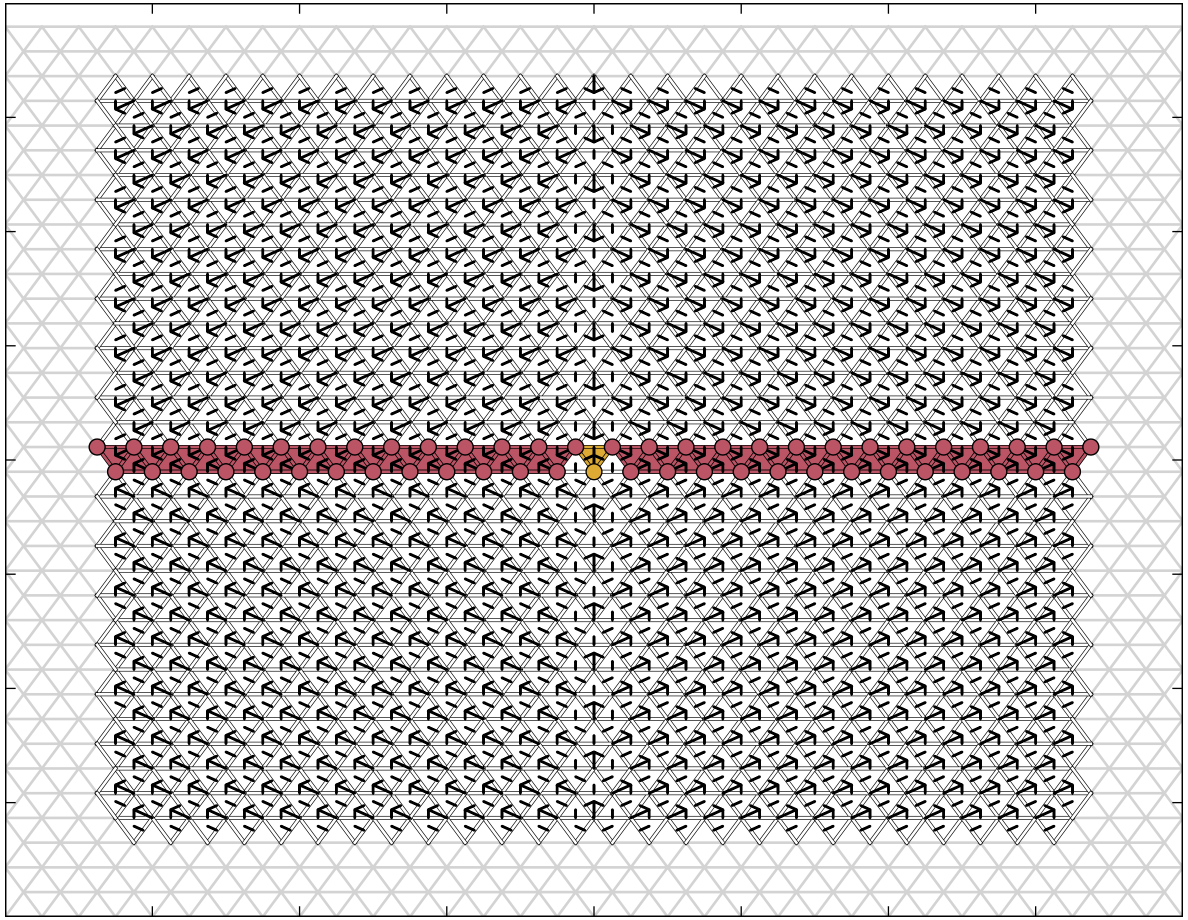
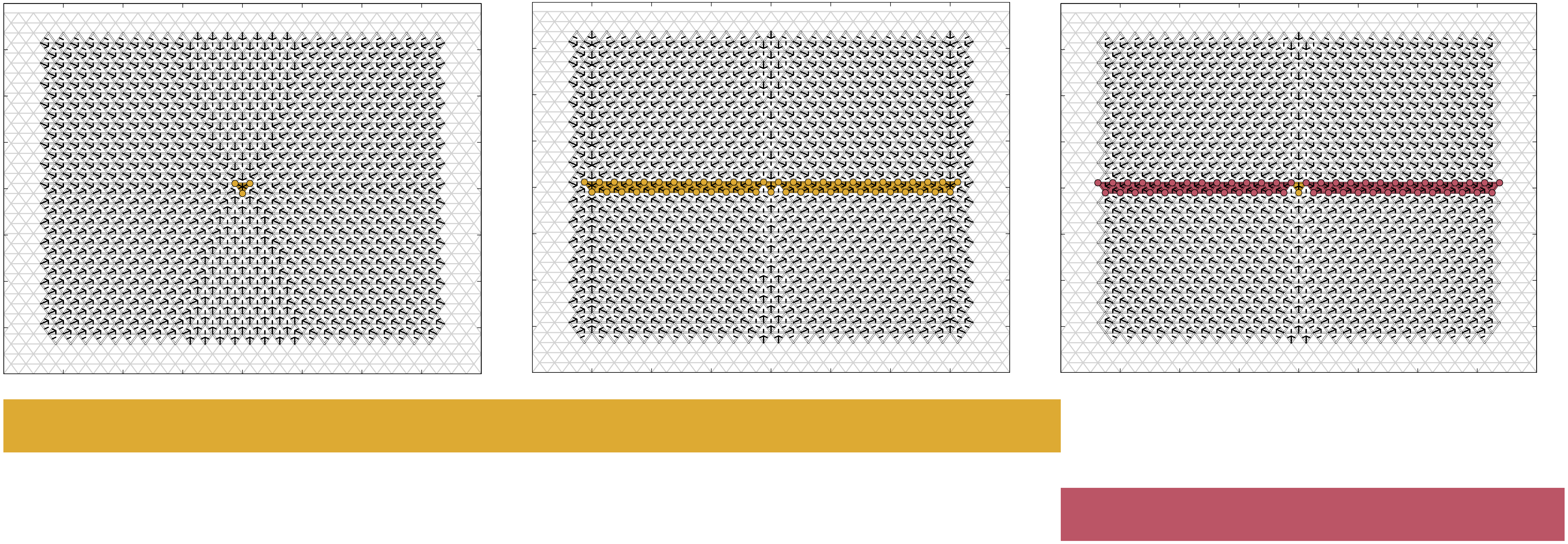


# Conley Index Persistence



# Conley Index Persistence: Variable N

Have a sequence of isolated invariant sets $A_1, A_2, \ldots, A_n$ isolated by $N_1, N_2, \ldots, N_n$ ($A_i$ isolated by $N_i$ and $N_{i-1}$).

$$(\mathsf{pf}_{N_i}(\mathsf{cl}(A_i)), \mathsf{pf}_{N_i}(\mathsf{mo}(A_i))) \supseteq (\mathsf{pf}_{N_i}(\mathsf{cl}(A_i)) \cap \mathsf{pf}_{N_i}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_i}(\mathsf{mo}(A_i)) \cap \mathsf{pf}_{N_i}(\mathsf{mo}(A_{i+1})))$$
$$\subseteq (\mathsf{pf}_{N_i}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_i}(\mathsf{mo}(A_{i+1})))$$

$$(\mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+1}))) \supseteq$$
$$(\mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+1})) \cap \mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+2})), \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+1})) \cap \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+2})))$$
$$\subseteq (\mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+2})), \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+2})))$$

# Conley Index Persistence: Variable N

How to connect the two index pairs?

$$(\mathsf{pf}_{N_i}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_i}(\mathsf{mo}(A_{i+1})))$$

$$(\mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+1})))$$

# Conley Index Persistence: Variable N

**<u>Theorem:</u>** If $(P_1, E_1)$, $(P_2, E_2)$ are strong index pairs for $A$ in $N_1, N_2$, where $A$ is isolated by $N_1, N_2, N_1 \cup N_2$, then $(\mathrm{pf}_{N_1 \cup N_2}(P_1 \cup P_2), \mathrm{pf}_{N_1 \cup N_2}(E_1 \cup E_2))$ is an index pair for $A$ in $N_1 \cup N_2$

# Conley Index Persistence: Variable N

Strategy replace

$$(\mathsf{pf}_{N_i}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_i}(\mathsf{mo}(A_{i+1})))$$

and

$$(\mathsf{pf}_{N_{i+1}}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_{i+1}}(\mathsf{mo}(A_{i+1})))$$

with

$$(\mathsf{pf}_{N_i \cup N_{i+1}}(\mathsf{cl}(A_{i+1})), \mathsf{pf}_{N_i \cup N_{i+1}}(\mathsf{mo}(A_{i+1})))$$

# Conley Index Persistence: Variable N

To summarize, given $A_1, A_2, \ldots, A_n$ isolated by $N_1, N_2, \ldots, N_n$ , where each $A_i$ is isolated by $N_i \cup N_{i+1}$, we obtain a zigzag filtration.

But given $A_1$ and $N_1$, how to find $A_2, \ldots, A_n$ and $N_2, \ldots, N_n$?

# Finding R

Strategy: Given $N_1, A_1$, define a "collar" $C$ around $A_1$, then find set of simplices $R$ so that $A_1$ is isolated by $N_1 \cup C \setminus R$.

$$N_2 := C \setminus R$$

Take $A_2$ to be the maximal invariant set in $N_2$

How to find $R$ ?

# Finding R

**FindR(**A, N, $\delta$):

$$A' = \mathsf{pb}_N(A) = \{\sigma \in N \mid A \cap \mathsf{pf}_N(\{\sigma\}) \neq \emptyset\}$$

$C'$ denotes a $\delta$-collar of A

**while** there exists a path in $C'$ from A to A':

Let $\sigma$ denote the last simplex not in A'

add all cofaces of $\sigma$ to R

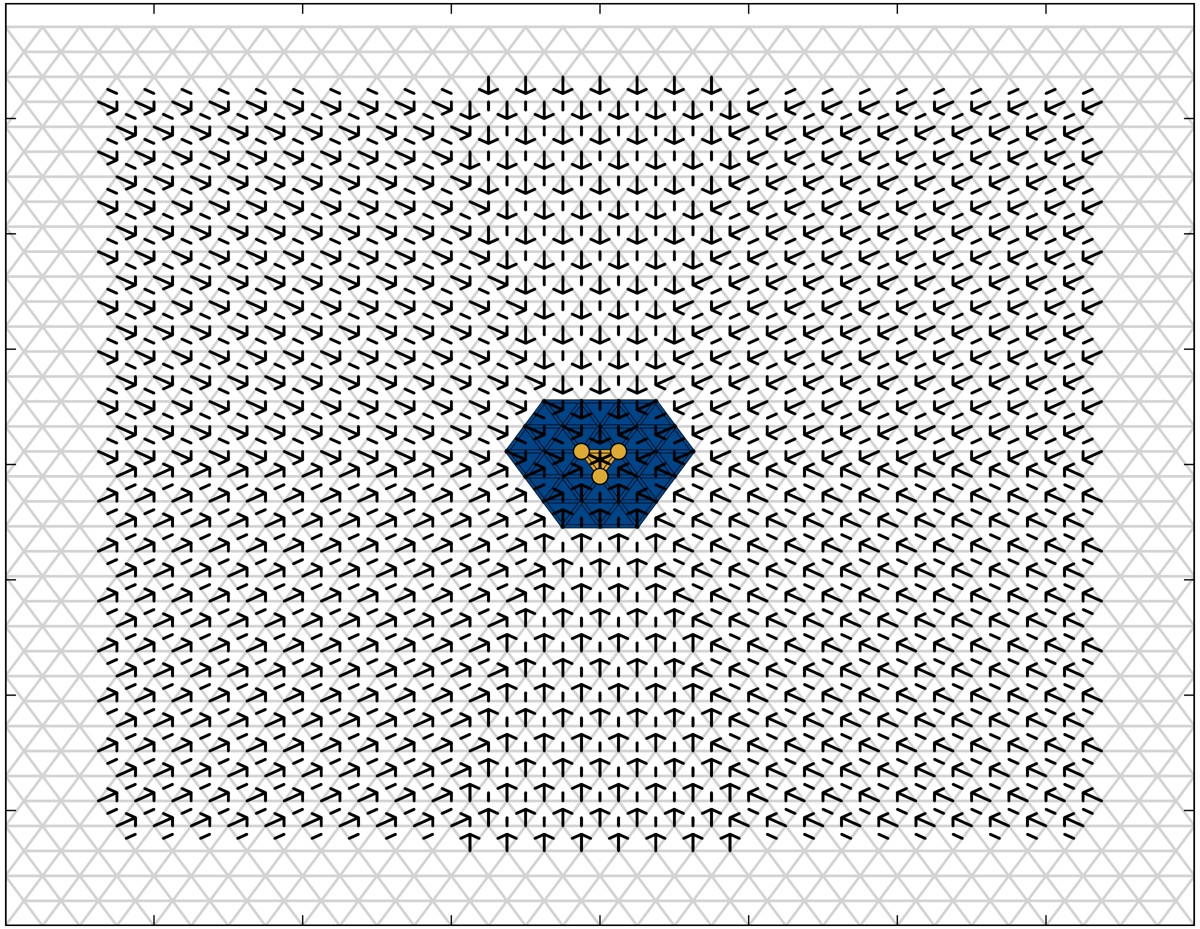
return R

# Finding R

**Theorem:** The set $C' \setminus R$ is closed.

**Theorem:** The invariant set $A$ is isolated by $N \cup (C' \setminus R)$

# Pitchfork Bifurcation: Variable N



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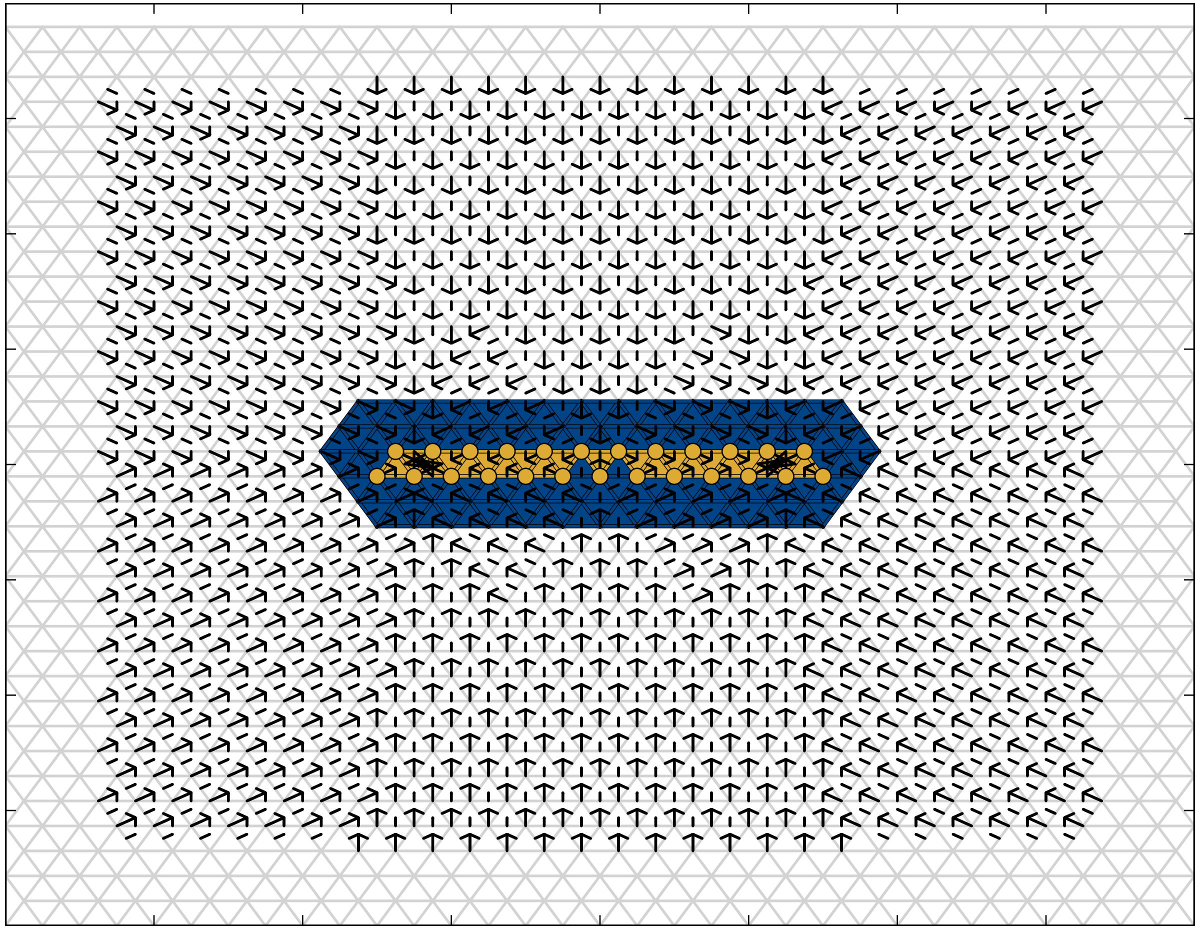

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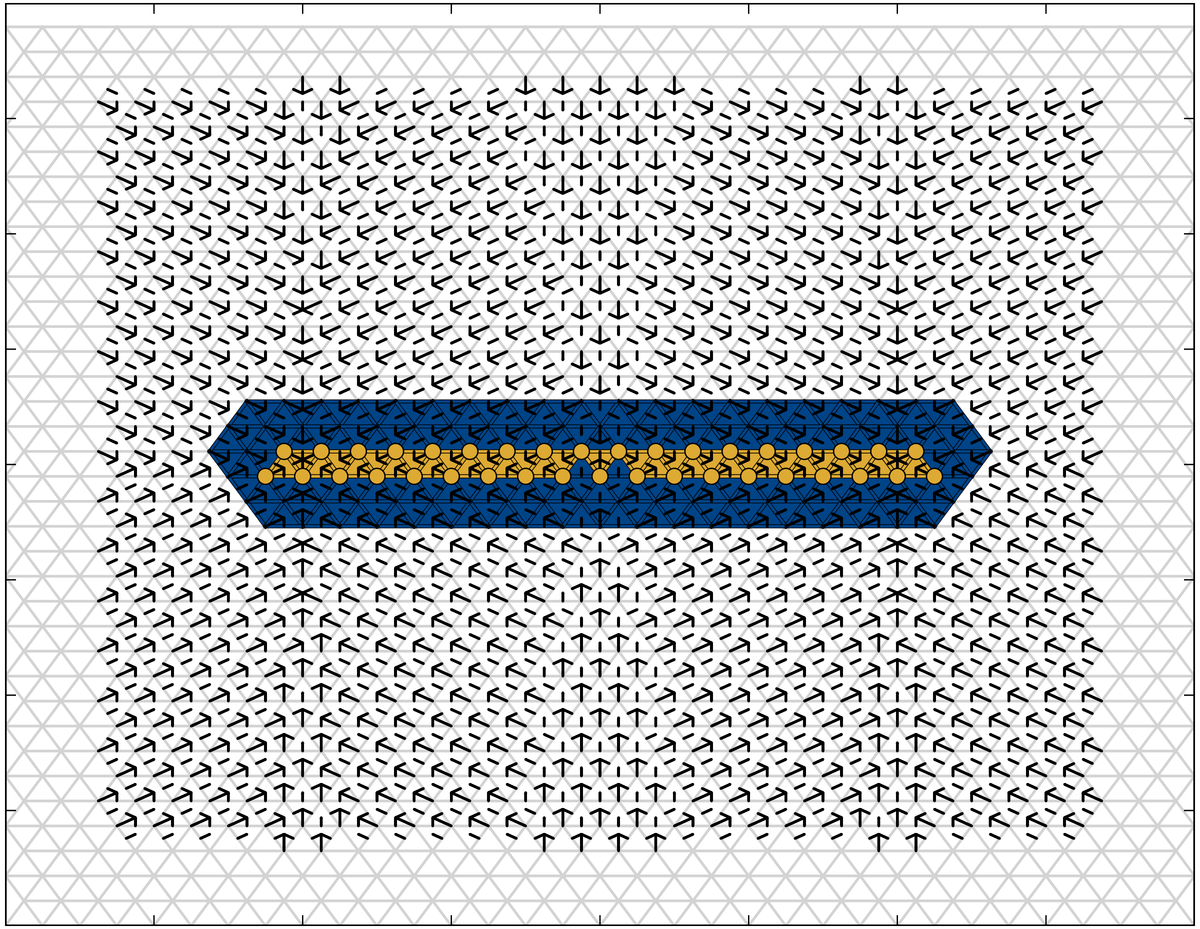

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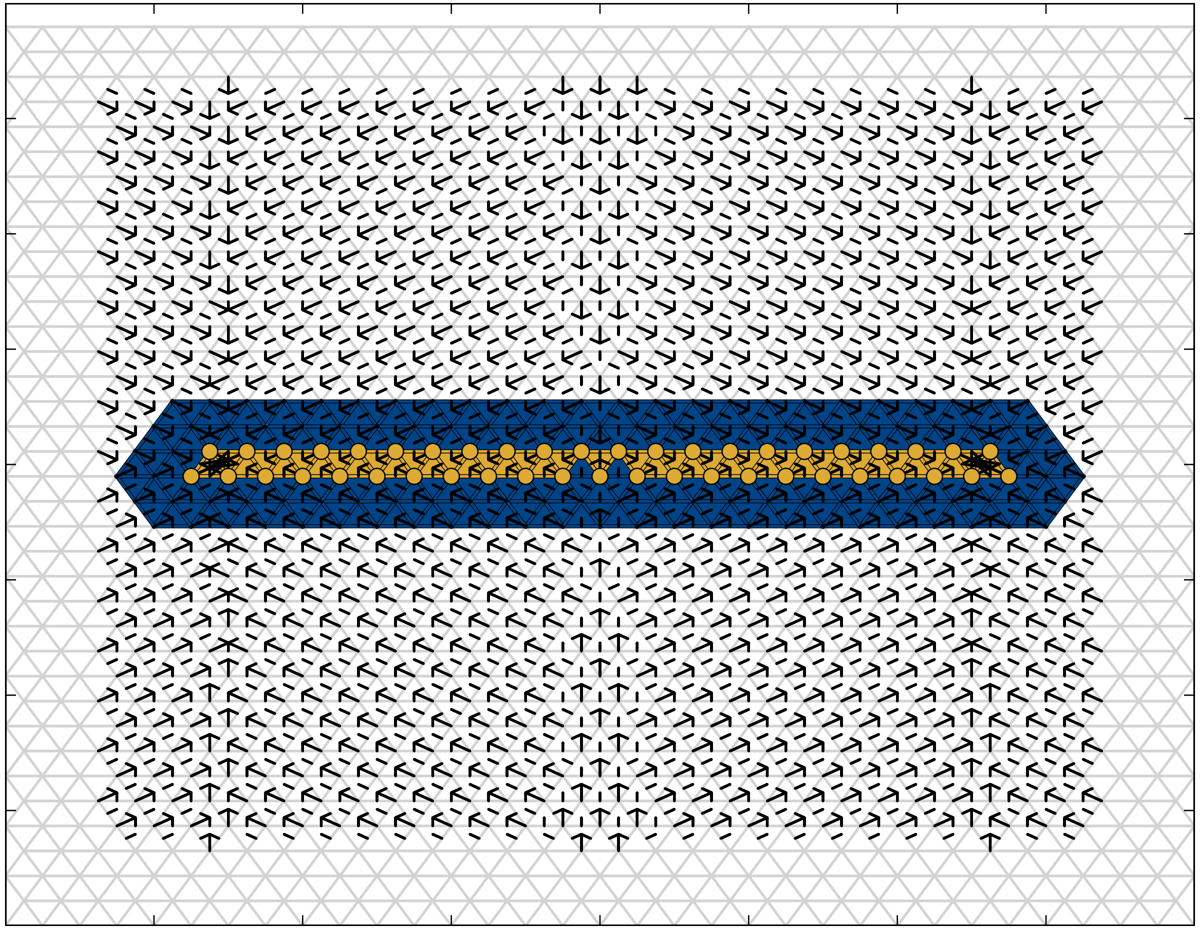

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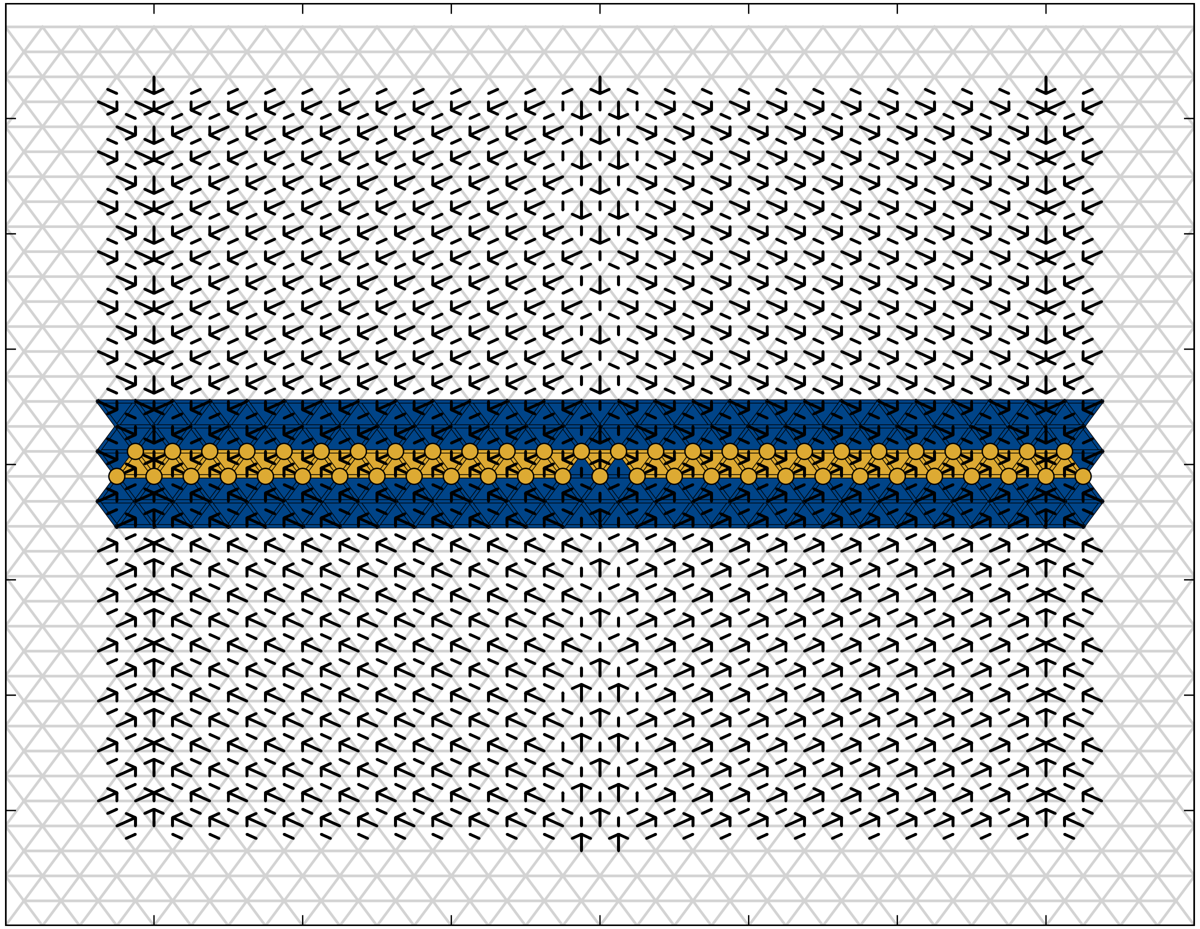

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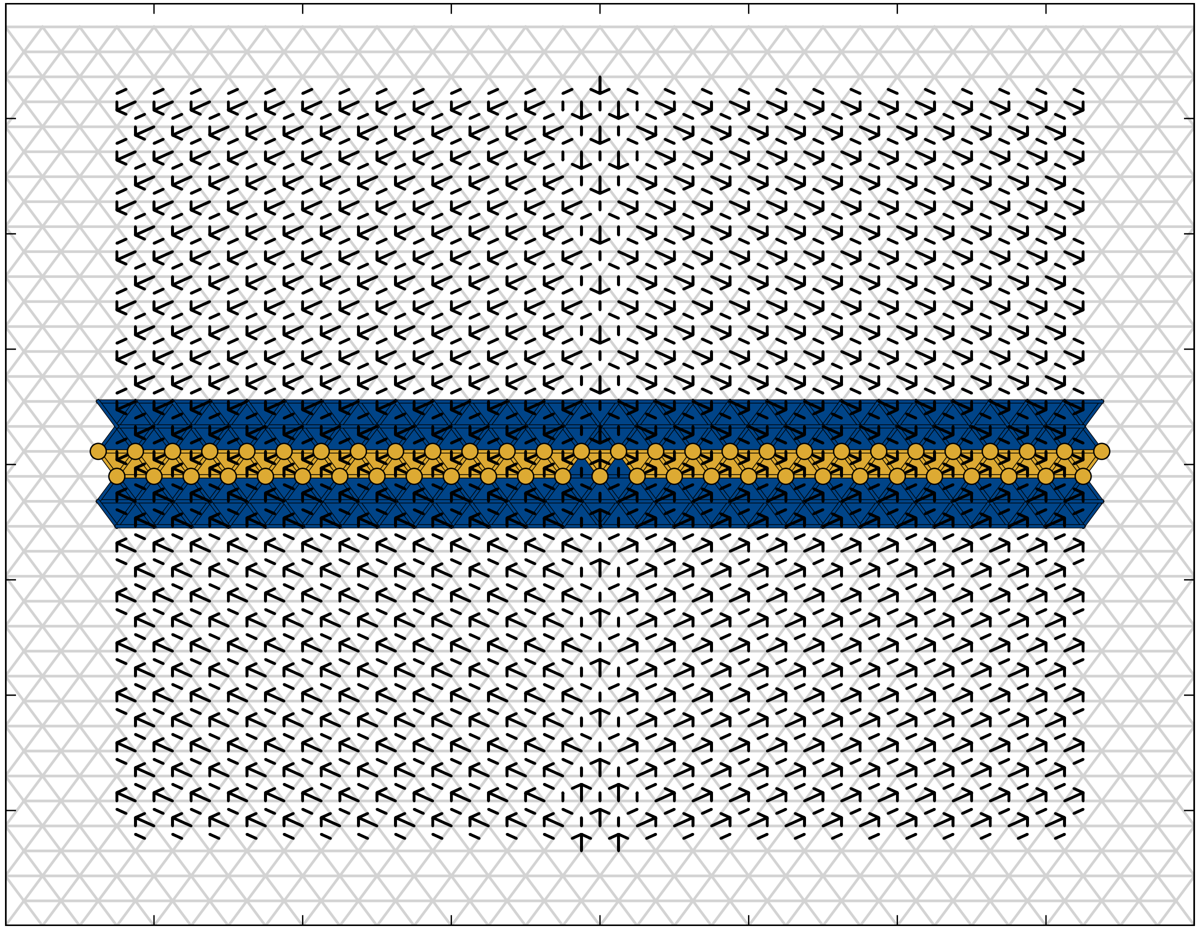

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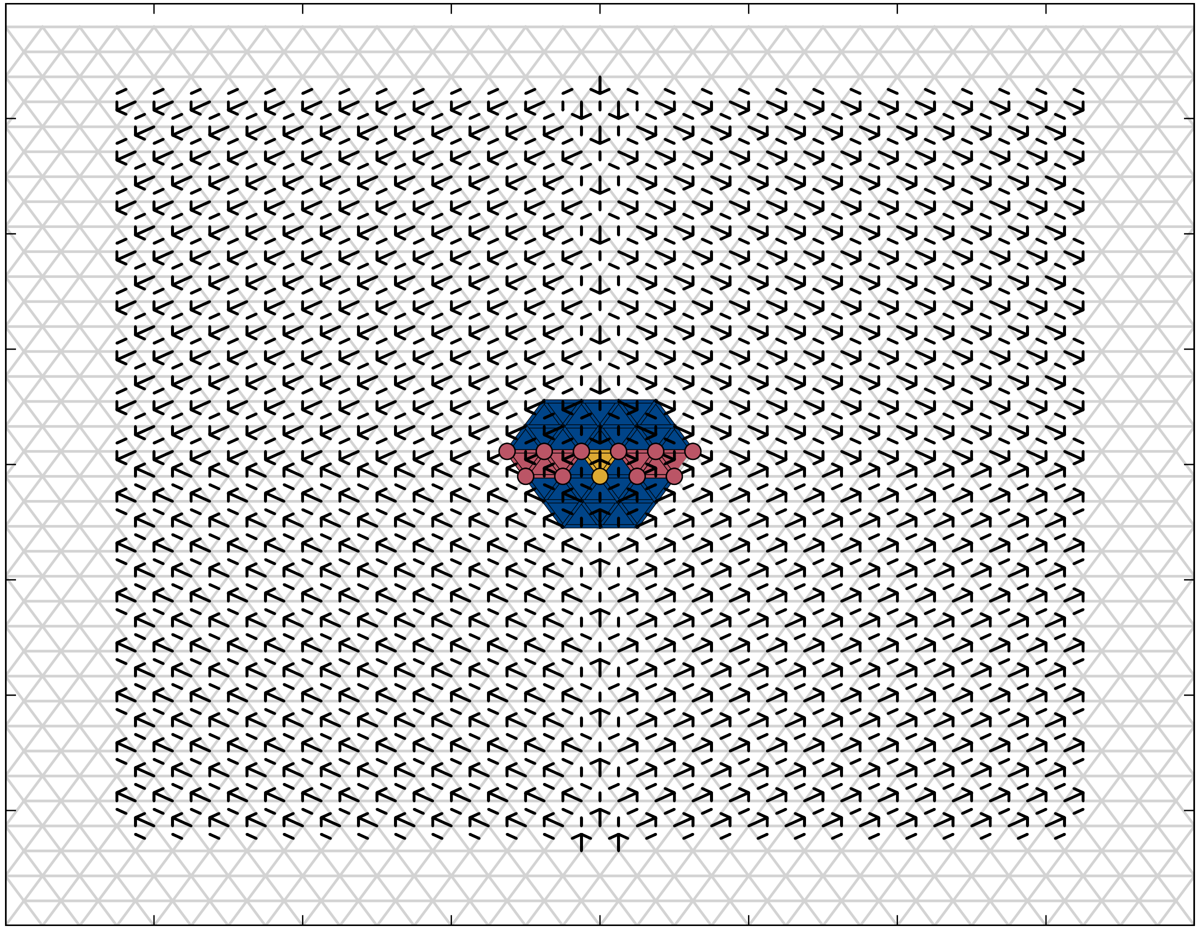

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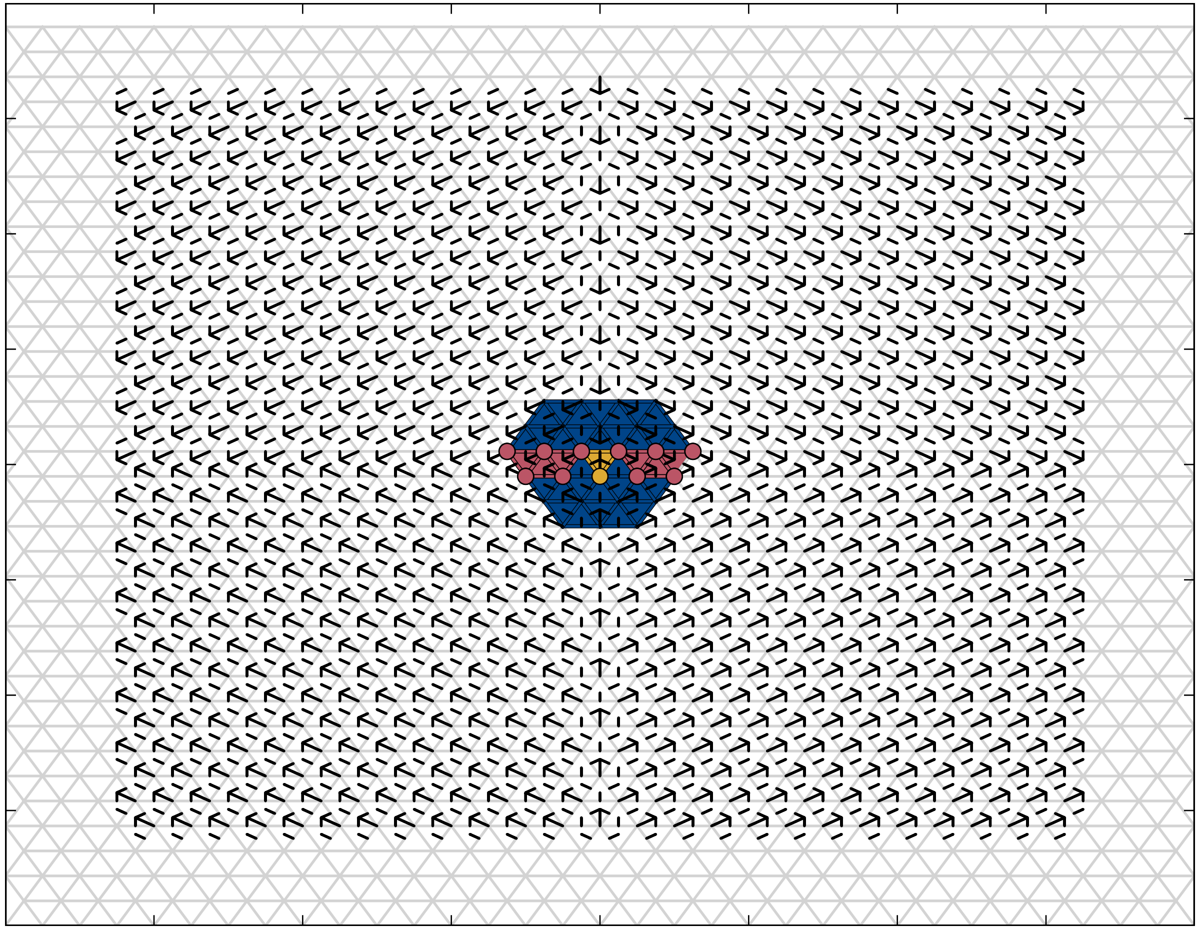

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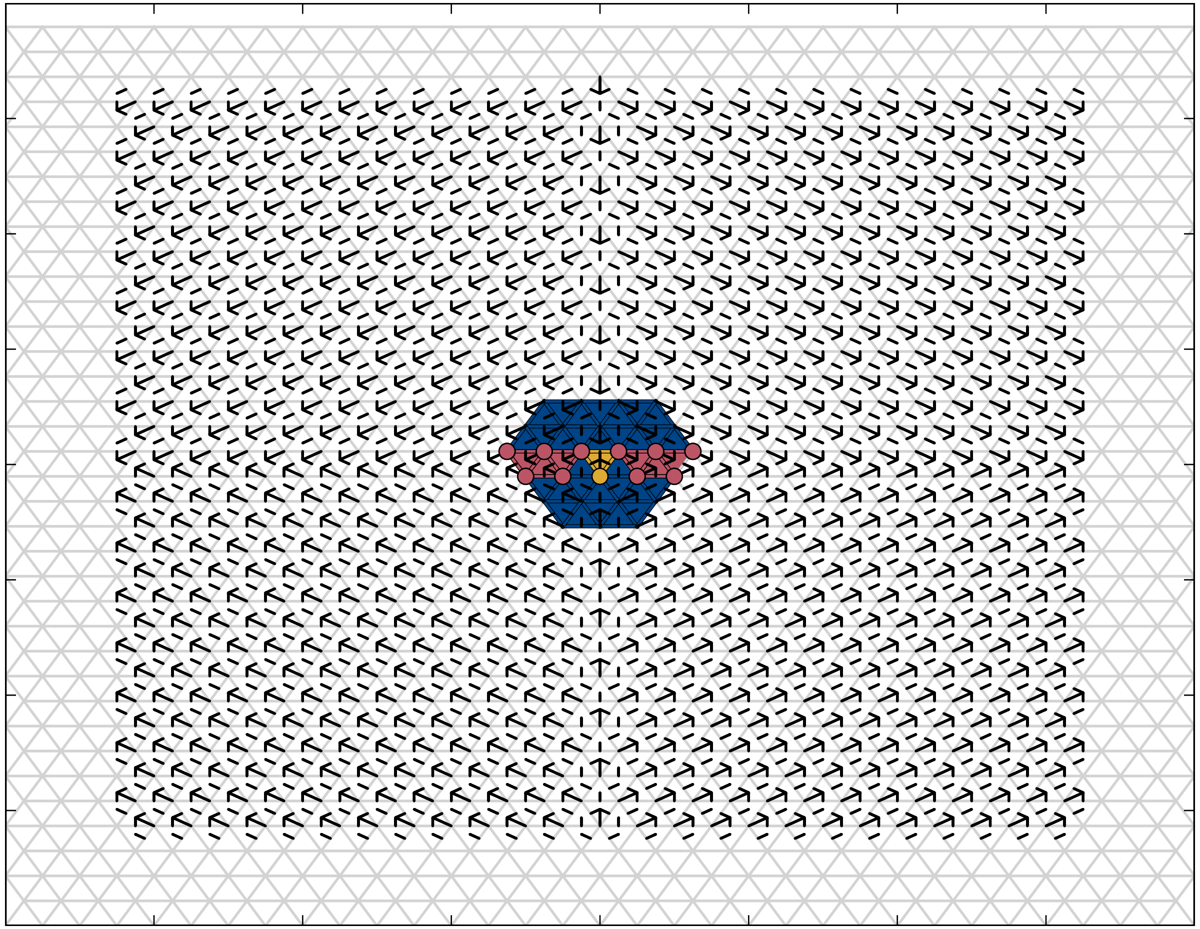

# Conclusion & Future Work

- Stability?

- Inference?

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